Chapter 2

Existence Theory and Properties of Solutions

This chapter contains some of the most important results of the course. Our first goal is to prove a theorem that guarantees the existence and uniqueness of a solution to an initial value problem on some, possibly small, interval. We then investigate the issue of how large this interval might be. The last section of the chapter provides some insight into how a solution of an initial value problem changes when the differential equation or initial conditions are altered.

2.1 Introduction

Consider an $n^{\text{th}}$ order differential equation in the form

$$y^{(n)} = g(t, y, y', y'', \ldots, y^{(n-1)}).$$

It is a standard practice to convert such an $n^{\text{th}}$ order equation into a first order system by defining

$$
\begin{align*}
x_1 &= y \\
x_2 &= y' \\
&\vdots \\
x_n &= y^{(n-1)}.
\end{align*}
$$

We will denote vectors in $\mathbb{R}^n$ by $x = (x_1, \ldots, x_n)$ so that our scalar equation is now represented in vector form as

$$
\frac{dx}{dt} = x'(t) = \begin{pmatrix} x_2 \\ x_3 \\ \vdots \\ g(t, x_1, x_2, \ldots, x_n) \end{pmatrix} = f(t, x(t)).
$$
Consequently it suffices to focus upon 1st ordinary differential equations denoted by

\[ x'(t) = f(t, x(t)) \]  \hspace{1cm} (2.1.1)

where \( x \in \mathbb{R}^n \) and \( f(t, x) \in \mathbb{R}^n \) is defined on an open set \( U \subseteq \mathbb{R} \times \mathbb{R}^n \). A solution of (2.1.1) is a differentiable function \( \xi : J \rightarrow \mathbb{R}^n \) where \( J \) is an open interval in \( \mathbb{R} \) and for \( t \in J \), \( (t, \xi(t)) \in U \), and

\[ \xi'(t) = f(t, \xi(t)). \]

A solution \( \xi(t) \) of the initial value problem (IVP)

\[ \begin{align*}
  x'(t) & = f(t, x(t)) \\
  x(t_0) & = x_0
\end{align*} \hspace{1cm} (2.1.2) \]

is a solution of the differential equation (2.1.1) that also satisfies the initial condition \( \xi(t_0) = x_0 \).

**Example 2.1.1**  
Recall from Example (1.2.3) that the IVP

\[ \begin{align*}
  x' & = \frac{x}{t} + t = f(t, x) \\
  x(0) & = x_0
\end{align*} \]

has infinitely many solutions if \( x_0 = 0 \) and no solution if \( x_0 \neq 0 \). This suggests that continuity of \( f(t, x) \) would be a minimal condition to ensure existence of a solution to an IVP.

**Example 2.1.2**  
Consider

\[ x' = f(t, x) = x^{1/3}. \]

By separation of variables we get the family of solutions

\[ \xi(t) = \left( \frac{2}{3} (t + c) \right)^{3/2} \]

Now consider the IVP

\[ \begin{align*}
  x' & = x^{1/3} \\
  x(0) & = 0
\end{align*} \]
For each $c > 0$ we obtain a solution $\xi_c$ where

\[ \xi_c(t) = \begin{cases} 
\left( \frac{2}{3} (t - c) \right)^{3/2}, & t \geq c \\
0, & t \leq c.
\end{cases} \]

Thus we see that continuity of $f(t, x)$ is not enough to ensure uniqueness.

Fig. 2.1.1. There are infinitely many solutions to the IVP in Example 2.1.2.

Our goal is to prove that under appropriate hypotheses on $f(t, x)$, the initial value problem (2.1.2) has a solution defined on an interval $(t_0 - \epsilon, t_0 + \epsilon)$ and any two such solutions must agree on their common domain. The above examples suggest that an appropriate notion of smoothness must be assumed of $f(t, x)$. To describe the regularity that will be required we need to introduce some terminology. For $x \in \mathbb{R}^n$ we denote the sup, or $l_\infty$ norm by

\[ |x| = \max_n \{|x_n|\}. \]

Let $(X, d)$ be a metric space and denote the open ball of radius $r$ around $x_0$ by

\[ B_r(x_0) = \{ x \mid d(x, x_0) < r \}. \]

$\overline{B}_r(x_0)$ will denote the closed ball $\{ x \mid d(x, x_0) \leq r \}$. Let $J_\epsilon(t) = (t - \epsilon, t + \epsilon) \subset \mathbb{R}$ and assume that $f(t, x) : U \subset \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$. The existence and uniqueness results we prove are obtained by assuming that $f(t, x)$ satisfies a Lipschitz condition. More precisely, we say that $f(t, x)$ is locally Lipschitz with respect to $x$ if for any $(t_0, x_0) \in U$ there exists $L \geq 0$ and $\epsilon > 0$, so that $J_\epsilon(t_0) \times B_\epsilon(x_0) \subset U$ and

\[ |f(t, x) - f(t, y)| \leq L|x - y|, \quad \text{for } t \in J_\epsilon(t_0) \text{ and } x, y \in B_\epsilon(x_0). \]

It is easily verified that if $f(t, x)$ is continuous and the partial derivatives $\partial f_i / \partial x_i$ exist and are continuous on $U$, then $f$ is locally Lipschitz with respect to the second variable. Previously we saw that the IVP of Example(2.1.2) has infinitely many solutions. Note that the function $f(x) = x^{1/3}$ is not Lipschitz at the origin.
The notion of a contractive mapping is central to many existence arguments. If $\alpha$ satisfies $0 < \alpha < 1$, and $T : X \to X$ is a mapping, we say $T$ is an $\alpha$-contraction if
\[
d(T(x), T(y)) \leq \alpha d(x, y) \text{ for all } x, y \in X.
\]
If $T(p) = p$, we call $p$ is a fixed point of $T$. We will denote iterates of $T$ by
\[
T^0(x) = x, \ T^1(x) = T(x), \ T^2(x) = T(T^1(x)), \ldots \ T^n(x) = T(T^{n-1}(x)).
\]

The following lemma is crucial.

**Lemma 2.1.1** [Contraction Mapping Lemma]. Let $(X, d)$ be a complete metric space and $T : X \to X$ an $\alpha$-contraction. Then $T$ has a unique fixed point $p$. In fact, for any $x \in X$, the iterates $T^n(x)$ converge to $p$.

**Proof.** Define $f : X \to [0, \infty)$ by $f(x) = d(T(x), x)$. In other words, $f(x)$ is the distance $T$ moves $x$. Note that $f(p) = 0$ if $T(p) = p$ and observe that $f$ is continuous. Indeed
\[
f(x) = d(x, T(x)) \\
\leq d(x, y) + d(y, T(y)) + (T(y), T(x)) \\
\leq d(x, y) + f(y) + \alpha d(x, y)
\]
and so
\[
f(x) - f(y) \leq (1 + \alpha)d(x, y).
\]
Interchanging $x$ and $y$ we see
\[
|f(x) - f(y)| \leq (1 + \alpha)d(x, y).
\]

There are two inequalities satisfied by $f$. First,
\[
f(T(x)) = d(T(T(x)), T(x)) \leq \alpha d(T(x), x) = \alpha f(x).
\]
For the second inequality note that for $x, y \in X$,
\[
d(x, y) \leq d(x, T(x)) + d(T(x), T(y)) + d(T(y), y) \\
\leq f(x) + \alpha d(x, y) + f(y)
\]
and so
\[
d(x, y) \leq \frac{f(x) + f(y)}{1 - \alpha}.
\]
Now let $x_0$ be any point in $X$ and $x_n = T^n(x_0)$. Then from (2.1.3)
\[ f(x_n) \leq \alpha^n f(x_0) \]
and so $f(x_n) \to 0$ as $n \to \infty$. It follows from (2.1.4) that for any $n, m$
\[ d(x_n, x_m) \leq \frac{f(x_n) + f(x_m)}{1 - \alpha}. \]
For $n, m$ sufficiently large we can make the right hand side as small as we like and hence
\{x_n\} is a Cauchy sequence. Since $X$ is complete, there exists a $p \in X$ such that $x_n \to p$. Because $f$ is continuous, $f(x_n) \to f(p)$, and so $f(p) = 0$, i.e., $p$ is a fixed point of $T$.

To show uniqueness suppose $q$ is another fixed point. Then $f(q) = 0$ and from (2.1.4) we see $d(p, q) = 0$.

2.2 Existence and Uniqueness of Solutions

It turns out that continuity of $f(t, x)$ is sufficient to guarantee existence of a solution to the IVP
\[ \begin{align*}
    x'(t) &= f(t, x(t)) \\
    x(t_0) &= x_0.
\end{align*} \]

This result is referred to as Peano’s Theorem. Example (2.1.2) in the previous section showed that we need additional hypotheses on $f(t, x)$ to ensure uniqueness. The condition we need is Lipschitz continuity.

The next theorem is a first form of our Existence and Uniqueness Theorem.

Theorem 2.2.1 Let $f : U \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, $U$ open and $f(t, x)$ continuous and locally Lipschitz with respect to the second variable. The following two statements hold.

1. Select $(t_0, x_0) \in U$. For all $\epsilon > 0$ sufficiently small there is a differentiable function
\[ \xi : (t_0 - \epsilon, t_0 + \epsilon) \to \mathbb{R}^n \]
such that
\[ \begin{align*}
    (t, \xi(t)) &\in U, \quad t \in J_\epsilon(t_0) \\
    \xi'(t) &= f(t, \xi(t)), \quad t \in J_\epsilon(t_0) \\
    \xi(t_0) &= x_0.
\end{align*} \] (2.2.1)
That is, $\xi$ is a solution of the initial value problem.

2. If $\xi_1 : J_{\epsilon_1}(t_0) \to \mathbb{R}^n$ and $\xi_2 : J_{\epsilon_2}(t_0) \to \mathbb{R}^n$ are two differentiable functions that satisfy (2.2.1), then $\xi_1$ and $\xi_2$ agree on some open interval around $t_0$. 


First we need the following lemma.

**Lemma 2.2.1** A function \( \xi : J_c(t_0) \to \mathbb{R}^n \) is differentiable and satisfies (2.2.1) if and only if for \( t \in J_c(t_0) \), \( (t, \xi(t)) \in U \), \( \xi \) is continuous, and satisfies

\[
\xi(t) = x_0 + \int_{t_0}^{t} f(s, \xi(s))ds, \quad t \in J_c(t_0). \tag{2.2.2}
\]

**Proof.** Let \( \xi \) be a differentiable function that satisfies (2.2.1). Since \( \xi'(t) = f(t, \xi(t)) \) and \( f \) is continuous, \( \xi' \) is continuous. Thus by the Fundamental Theorem of Calculus

\[
\int_{t_0}^{t} \xi'(s)ds = \xi(t) - \xi(t_0) = \xi(t) - x_0 = \int_{t_0}^{t} f(s, \xi(s)) ds
\]

and so \( \xi(t) \) satisfies (2.2.2).

Conversely suppose \( \xi \) satisfies the conditions of the second part of the lemma. Then clearly

\[
\xi(t_0) = x_0
\]

and by the Fundamental Theorem of Calculus,

\[
\xi'(t) = f(t, \xi(t)).
\]

Thus \( \xi \) is differentiable and satisfies the IVP.

The proof of Theorem (2.2.1) is based on the Contractive Mapping Lemma where the underlying metric space will be a closed subset of \( BC(J_c(t_0); \mathbb{R}^n) \), the space of bounded continuous functions

\[
\xi : J_c(t_0) \to \mathbb{R}^n
\]

where for \( \xi_1, \xi_2 \in BC(J_c(t_0)) \),

\[
d(\xi_1, \xi_2) = \sup_{t \in J_c(t_0)} \{||\xi_1 - \xi_2||(t)\}
\]

\[
= ||\xi_1 - \xi_2||.
\]

**Proof of Theorem 2.2.1.** Since \( f \) is locally Lipschitz with respect to the second variable, we can find an \( r > 0 \) such that \( [t_0 - r, t_0 + r] \times \overline{B}_r(x_0) \subset U \) and

\[
|f(t, x) - f(t, y)| \leq L|x - y| \text{ for all } (t, x), (t, y) \in [t_0 - r, t_0 + r] \times \overline{B}_r(x_0).
\]
2.2. EXISTENCE AND UNIQUENESS OF SOLUTIONS

Since \([t_0 - r, t_0 + r] \times \overline{B}_r(x_0)\) is compact and \(f\) is continuous, there exists an \(M\) for which

\[|f(t, x)| \leq M \text{ for all } (t, x) \in [t_0 - r, t_0 + r] \times \overline{B}_r(x_0).\]

Choose \(\epsilon > 0\) so small such that

\[
\begin{align*}
\epsilon &< r \\
\epsilon M &< r \\
\epsilon L &< 1.
\end{align*}
\]

Let \(X \subset BC(J_\epsilon(t_0); \mathbb{R}^n)\) be the space of continuous functions

\[\xi : J_\epsilon(t_0) \to \overline{B}_r(x_0).\]

Then \(X\) is a closed subset of \(BC(J_\epsilon(t_0); \mathbb{R}^n)\) and hence is complete. Note that if \(\xi \in X, t \in J_\epsilon(t_0)\) and \(\epsilon < r\), then we certainly have \((t, \xi(t)) \in J_\epsilon(t_0) \times \overline{B}_r(x_0) \subseteq U\). For \(\xi \in X\), define \(T\xi\) on \(J_\epsilon(t_0)\) by

\[T\xi(t) = x_0 + \int_{t_0}^{t} f(s, \xi(s))ds.\]

By the Fundamental Theorem of Calculus \(T\xi\) is continuous and

\[
|T\xi(t) - x_0| = \left| \int_{t_0}^{t} f(s, \xi(s))ds \right| \leq \int_{t_0}^{t} |f(s, \xi(s))|ds \\
\leq M|t - t_0| < \epsilon M < r.
\]

Hence \(T\xi(t) \in B_r(x_0)\) and so \(T\xi \in X\). Thus \(T : X \to X\).

We now show \(T\) is a contraction. If \(\xi, \zeta \in X\),

\[
|T\xi(t) - T\zeta(t)| = |x_0 + \int_{t_0}^{t} f(s, \xi(s))\, ds - x_0 - \int_{t_0}^{t} f(s, \zeta(s))\, ds| \\
\leq \int_{t_0}^{t} |f(s, \xi(s)) - f(s, \zeta(s))|\, ds \\
\leq L\int_{t_0}^{t} |\xi(s) - \zeta(s)|\, ds \\
\leq L|t - t_0| ||\xi - \zeta|| \\
= L|t - t_0| ||\xi - \zeta|| \\
\leq \epsilon L||\xi - \zeta||.
\]
Thus

$$\sup_{t \in J(t_0)} |T\xi(t) - T\zeta(t)| = ||T\xi - T\zeta|| \leq \epsilon L||\xi - \zeta||$$

and since $\epsilon L < 1$, $T$ is a contraction. Hence $T$ has a fixed point and so there exists $\xi \in X$ such that

$$T\xi(t) = \xi(t) = x_0 + \int_{t_0}^{t} f(s, \xi(s))ds.$$  

By Lemma (2.2.1), this $\xi(t)$ is a solution (2.2.1).

Before proceeding to the proof of the second statement, note that given $(t_0, x_0)$ we first choose $r$ such that $K_r = [t_0 - r, t_0 + r] \times \overline{B}_r(x_0) \subset U$. Once we select $\epsilon$ so that $\epsilon < r$, $\epsilon M < r$, $\epsilon L < 1$, we can consider the set $X \subseteq BC(J_\epsilon(t_0); \mathbb{R}^n)$ in which $T$ has a fixed point. In this sense the set $X$ may be regarded as a one-parameter family $X(\epsilon)$ and the fixed point, though unique in $X(\epsilon)$, does depend on $\epsilon$.

To prove the second statement of the proposition suppose $\xi_1, \xi_2$ are two solutions of the IVP. The intersection of their domains is an open interval, say $(t_0 - \epsilon, t_0 + \epsilon)$. Since $\xi_1(t_0) = \xi_2(t_0) = x_0$, and $\xi_1, \xi_2$ are continuous we can select $\epsilon$ such that $\xi_1, \xi_2 : J_\epsilon(t_0) \rightarrow \overline{B}_r(x_0)$. We can further decrease $\epsilon$ if necessary so that $\epsilon < r$, $\epsilon M < r$ and $\epsilon L < 1$. With this choice of $\epsilon$, we then get that $\xi_1, \xi_2 \in X(\epsilon)$ and since $T : X(\epsilon) \rightarrow X(\epsilon)$ has a unique fixed point, $\xi_1(t) = \xi_2(t)$, $t \in J_\epsilon(t_0)$.

In summary, we have that for $(t_0, x_0) \in U$ there exists $\epsilon > 0$ such that the IVP has a solution $\xi$ that is defined on $(t_0 - \epsilon, t_0 + \epsilon)$ if $\epsilon < r$, $\epsilon M < r$, $\epsilon L < 1$. Note that in showing $T\xi(t) \in \overline{B}_r(x_0)$ we showed all iterates satisfy

$$|T\xi - x_0| \leq M|t - t_0| < \epsilon M.$$  

In particular the graph of the solution to (2.2.1) lies in the region $\mathbb{R}$ as depicted in the figure below. Note that if $M$ is large, the graph of the solution may escape the set $K_r$ unless the domain of the solution is restricted as required by the condition $\epsilon M < r$. 


An improved statement of Theorem 2.2.1 constitutes our main Existence and Uniqueness Theorem. Note that this a ‘local’ result in that the time interval on which the solution exists may be small.

**Theorem 2.2.2** [Existence and Uniqueness] Assume $f : U \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and locally Lipschitz with respect to the second variable. If $(t_0, x_0) \in U$, then there is an $\epsilon > 0$ such that the IVP

$$
\begin{align*}
  x' &= f(t, x) \\
  x(t_0) &= x_0
\end{align*}
$$

has a unique solution on the interval $(t_0 - \epsilon, t_0 + \epsilon)$.

**Proof** We know that for all sufficiently small $\epsilon$, the initial value problem has a solution. We need only prove that if $\xi_1, \xi_2$ satisfy the IVP on $J_\epsilon(t_0)$, then $\xi_1 = \xi_2$ on $J_\epsilon(t_0)$.

Let $S = \{ t \in (t_0 - \epsilon, t_0 + \epsilon) | \xi_1(t) = \xi_2(t) \}$. $S$ is not empty since $\xi(t_0) = \xi_2(t_0)$. Since $\xi_1$ and $\xi_2$ are continuous, $S$ is closed in $J_\epsilon(t_0)$. Let $\hat{t} \in S$ and $\hat{x} = \xi_1(\hat{t}) = \xi_2(\hat{t})$. Then $\xi_1, \xi_2$ solve the IVP with initial condition $(\hat{t}, \hat{x})$. By the previous proposition, $\xi_1$ and $\xi_2$ agree on an open interval $J \subseteq S$ containing $\hat{t}$. Hence $S$ is open and closed and since $J_\epsilon(t_0)$ is connected, $S = J_\epsilon(t_0)$.

The proof of Theorem 2.2.1 can be used to obtain a sequence of approximations that converge to the solution of the IVP. It is customary to begin the iteration process with
$x(t) = x_0$. Then

$$x_1(t) = Tx = x_0 + \int_{t_0}^{t} f(s, x_0) ds$$

$$x_2(t) = T^2 x = x_0 + \int_{t_0}^{t} f(s, x_1(s)) ds$$

$$\vdots \quad \vdots$$

$$x_n(t) = T^n x = x_0 + \int_{t_0}^{t} f(s, x_{n-1}(s)) ds.$$ 

From our results we know that $\{x_n(t)\}$ converges to a solution of the IVP in some neighborhood of $t_0$. This sequence of approximate solutions are known as Picard iterates. The usefulness of approximating a solution by this procedure has been somewhat enhanced by the availability of computer algebra systems such and Maple and Mathematica.

### 2.3 Continuation and Maximal Intervals of Existence

Our existence theorem is of local nature in that it provides for the existence of a solution to the IVP

$$x'(t) = f(t, x(t))$$

$$x(t_0) = x_0$$

defined in an interval $(t_0 - \epsilon, t_0 + \epsilon)$

**Example 2.3.1** The solution of

$$x' = x^2$$

$$x(0) = 1$$

is

$$x(t) = \frac{1}{1 - t}$$

Here $U = \mathbb{R} \times \mathbb{R}$. Note that the solution is defined for $-\infty < t < 1$. As $t \to 1^-$, the graph of the $x(t)$ leaves every closed and bounded subset of $U$. We will prove a theorem that reflects this general behavior. That is, the solution of an IVP can be defined on an interval $(m_1, m_2)$ where either $m_2 = +\infty$ or the graph of the solution escapes every closed and bounded subset of $U$ as $x \to m_2$ (and similarly for $m_1$).
2.3. CONTINUATION AND MAXIMAL INTERVALS OF EXISTENCE

Throughout this section we suppose $f : U \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, $U$ open, and $f(t, x)$ is continuous on $U$ and locally Lipschitz in $x$. Suppose $\xi(t)$ is a solution of

$$
\begin{align*}
    x' &= f(t, x) \\
    x(t_0) &= x_0
\end{align*}
$$

that is defined for $\gamma < t < \delta$ and $(\delta, \xi(\delta^-)) \in U$. Now consider the IVP

$$
\begin{align*}
    x' &= f(t, x) \\
    x(\delta) &= \xi(\delta^-).
\end{align*}
$$

We know this problem has a solution, say $\psi(t)$, defined on $\delta \leq t < \delta + \epsilon$. Define

$$
y(t) = \begin{cases} 
    \xi(t) & \gamma < t < \delta \\
    \psi(t) & \delta \leq t < \delta + \epsilon.
\end{cases}
$$

Clearly $y(t)$ is continuous. Moreover,

$$
y(t) = \xi(\delta^-) + \int_\delta^t f(s, \psi(s)) \, ds \text{ for } \delta < t < \delta + \epsilon
$$

and

$$
\xi(\delta^-) = x_0 + \int_{x_0}^\delta f(s, \xi(s)) \, ds.
$$

Hence

$$
y(t) = x_0 + \int_t^\delta f(s, \xi(s)) \, ds + \int_\delta^t f(s, \psi(s)) \, ds
$$

or

$$
y(t) = x_0 + \int_t^{t_0} f(s, y(s)) \, ds, \quad \delta \leq t < \delta + \epsilon.
$$

Since we clearly have

$$
y(t) = x_0 + \int_{t_0}^t f(s, y(s)) \, ds, \quad \gamma < t < \delta,
$$

it follows from Lemma(2.2.1) that

$$
\begin{align*}
    y'(t) &= f(t, y(t)), \quad \gamma < t < \delta + \epsilon \\
    y(t_0) &= x_0
\end{align*}
$$

and so $y(t)$ is a solution of the IVP that is defined on a larger interval.

The above process is referred to as continuation to the right. In the same way one could construct a continuation to the left. By our uniqueness result any extension of the solution
from \((\gamma, \delta)\) to \((\gamma - \epsilon_1, \delta + \epsilon_2)\) is unique. The geometric interpretation of the continuation process is displayed in Figure 2.3.1.

![Fig. 2.3.1. The continuation process.](image)

**Definition (2.3.1)** Let \(\xi\) be a solution of an ordinary differential equation on an interval \(J\). A function \(\tilde{\xi}\) is called a continuation of \(\xi\) if

1. \(\tilde{\xi}\) is defined on an interval \(\tilde{J}\) where \(J \subset \tilde{J}\).
2. \(\tilde{\xi} = \xi\) for \(t \in J\), and
3. \(\tilde{\xi}\) satisfies the ordinary differential equation on \(\tilde{J}\).

**Theorem 2.3.1** Assume \(f : U \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n\), \(U\) open and \(f(t, x)\) continuous and locally Lipschitz with respect to the second variable. Then there exists a solution \(\xi(t)\) of the IVP

\[
\begin{align*}
x' &= f(t, x) \\
x(t_0) &= x_0
\end{align*}
\]

defined on an interval \((m_1, m_2)\) with the property that if \(\psi\) is any other solution of the IVP, the domain of \(\psi\) is contained in \((m_1, m_2)\).

**Proof** Let \(M\) denote the set of all intervals on which solutions of the IVP are defined. That \(M\) is not empty follows from the Existence Theorem. Let \(M_1\) be the set of all right hand endpoints of \(M\) and \(M_2\) the set of all left hand endpoints. Take

\[m_1 = \inf M_1, \quad m_2 = \sup M_2.\]
Pick any \( \hat{t} \in (m_1, m_2) \). Then there exists a solution of the IVP whose interval of definition includes \( \hat{t} \), say \( \xi \). Define a solution \( \xi(t) \) on \((m_1, m_2)\) by setting \( \xi(\hat{t}) = \xi(\hat{t}) \). By uniqueness it follows that \( \xi(t) \) is well defined and is a solution for all \( t \in (m_1, m_2) \).

The interval \((m_1, m_2)\) is called the maximal interval of existence corresponding to \((t_0, x_0)\). Furthermore, the maximal interval must be open (verify this).

**Example 2.3.2** Take \( U \) to be the right half plane and consider

\[
x'(t) = \frac{1}{t^2} \cos\left(\frac{1}{t}\right)
\]

\[
x(t_0) = x_0
\]

Then \( x(t) = c - \sin\left(\frac{1}{t}\right) \) and the IVP can be solved for any initial condition \((t_0, x_0)\), \( t_0 > 0 \). Note that the maximal interval of existence is \((0, \infty)\) and \( \lim_{t \to 0^+} x(t) \) does not exist.

**Example 2.3.3** Consider

\[
x'(t) = -3t^{4/3} \sin(t)
\]

\[
x(t_0) = x_0
\]

Solutions are \( x(t) \equiv 0 \) and \( x(t) = (c - \cos(t))^{-3} \) where \( c \) is determined by the initial data \((t_0, x_0)\). Nontrivial solutions are defined on \((-\infty, \infty)\) only if \(|c| > 1\). Thus, the maximal interval of existence may depend on the initial conditions. Moreover, this example and Example(2.3.1) suggest that the graph of a solution tends to infinity at a finite endpoint of the maximal interval of existence. This is indeed the case when \( f(t, x) \) is bounded, but the complete story is a bit more involved. The next few theorems address this issue and clarify these suggestions.

**Theorem 2.3.2** Assume \( f : U \subseteq \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n \), \( U \) open and \( f(t, x) \) continuous and locally Lipschitz with respect to the second variable and bounded on \( U \). If \( \xi(t) \) is a solution of the IVP,

\[
x'(t) = f(t, x)
\]

\[
x(t_0) = x_0
\]

and defined for \( \gamma < t < \delta \), then the limits

\[
\lim_{t \to \gamma^+} \xi(t), \quad \lim_{t \to \delta^-} \xi(t)
\]

exist. If \((\delta, \xi(\delta^-)), (\gamma, \xi(\gamma^+)) \in U\), then the solution can be extended to the right and left.
**Chapter 2. Existence Theory and Properties of Solutions**

**Proof** Let $t_1, t_2 \in (\gamma, \delta)$. Then

$$|\xi(t_1) - \xi(t_2)| \leq \int_{t_2}^{t_1} |f(s, \xi(s))| ds$$

$$\leq B|t_1 - t_2|.$$ 

If we pick $\{t_n\}$ such that $t_n \to \delta^-$, then for any $\epsilon > 0$,

$$|\xi(t_n) - \xi(t_m)| \leq B|t_n - t_m| < \epsilon$$

for all $n, m$ sufficiently large. Hence $\{\xi(t_n)\}$ is Cauchy and so converges. Thus $\lim_{n \to \infty} \xi(t_n)$ exists. An identical argument applies for $\lim_{t \to \delta^-} \xi(t)$.

The second assertion follows immediately from the remarks preceding the definition of continuation.

Compare this theorem with the result of Example (2.3.2) in which $f(t, x) = \frac{1}{t^2} \cos(\frac{1}{t})$ was not bounded on $U$. As we observed, the solution did not have a limit at the left hand endpoint of its maximal interval of existence.

**Theorem 2.3.3** Assume $f : U \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, $U$ open and $f(t, x)$ continuous and locally Lipschitz with respect to the second variable and bounded on $U$. Let $(m_1, m_2)$ denote the maximal interval of existence of the solution $\xi$ of the IVP

$$x' = f(t, x)$$

$$x(t_0) = x_0.$$ 

Then either $m_2 = \infty$ or $(m_2, \xi(m_2^-))$ is on the boundary of $U$. A similar statement holds for $m_1$.

**Proof.** First suppose $m_2 < \infty$ were finite. From the previous theorem, $\xi(m_2^-)$ exists and if $(m_2, \xi(m_2^-)) \in U$ then the solution could be extended to the right. It must follow that $(m_2, \xi(m_2^-))$ lies on the boundary of $U$. Similarly for $m_1$.

**Example 2.3.4** Reconsider the example

$$x' = x^2$$

$$x(0) = 1.$$ 

Here $U = \mathbb{R}^2$ and $\xi(t) = \frac{1}{1-t}$. Define

$$U_A = \{(t, x) \mid |t| < \infty, |x| < A\}.$$
The maximal interval of existence is \((m_1, m_2) = (-\infty, 1)\) and as \(t \to m_2^-\) the graph of the solution will always meet the boundary of \(U_A\) when \(t = 1 - 1/A\).

In general suppose \(f(t, x)\) is is continuous and locally Lipschitz with respect to the second variable on all of \(\mathbb{R} \times \mathbb{R}^n\) and the solution of an IVP has a maximal interval of existence, \((m_1, m_2)\) where \(m_2 < \infty\). One may modify the ideas in the previous example and apply Theorem(2.3.2) to conclude that as \(t \to m_2^-\) the graph of the solution always meets the boundary \(|x| = A\) of the set \(U_A\). Since \(A\) can be arbitrarily large, the following theorem must follow. (The details are left as an exercise.)

**Corollary 2.3.1** Let \(U = \mathbb{R} \times \mathbb{R}^n\) and \((m_1, m_2)\) denote the maximal interval of existence of the IVP. If \(|m_2| < \infty\), then
\[
\lim_{t \to m_2^-} |\xi(t)| = \infty.
\]
(Similarly for \(m_1\)).

This corollary provides a method for determining when a solution is global, that is, defined for all time \(t\). In particular, if \(f(t, x)\) is defined on all of \(\mathbb{R} \times \mathbb{R}^n\), then a solution is global if it does not blow up in finite time. These ideas are illustrated in the next examples.

**Example 2.3.5** Consider the equation for the damped, nonlinear pendulum.
\[
y''(t) + \alpha y' + \sin y = 0, \quad \alpha > 0
\]
\[
y(0) = y_0, \quad y'(0) = v_0.
\]
Rewrite the problem as a first order system,
\[
x_1 = y
\]
\[
x_2 = y'.
\]
Then
\[
x' = \frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -\alpha x_2 - \sin x_1 \end{pmatrix} = f(x)
\]
\[
x(0) = \begin{pmatrix} y_0 \\ v_0 \end{pmatrix}.
\]
Since \(\partial f_i/\partial x_j\) are continuous for all \((x_1, x_2)\), \(f\) is locally Lipschitz. Hence for any initial conditions the IVP has a unique solution. We now show the solution is global, i.e., it exists for all \(t\).
In a standard way, one first multiplies the equation by $y'$ to get

$$y'(y'' + \alpha y' + \sin y) = 0$$

and

$$\frac{d}{dt} \left( \frac{1}{2} (y')^2 - \cos y \right) = -\alpha (y')^2 \leq 0$$

or

$$\frac{d}{dt} \left( \frac{1}{2} (y')^2 + (1 - \cos y) \right) \leq 0.$$  

Thus

$$\frac{1}{2} (y'(t))^2 + (1 - \cos y(t)) \leq \frac{1}{2} (y'(0))^2 + (1 - \cos y(0)) = \frac{1}{2} (v_0)^2 + (1 - \cos y_0).$$

Let

$$1 - \cos y_0 + \frac{1}{2} (v_0)^2 = \frac{1}{2} p_0^2$$

and since $(1 - \cos y) \geq 0$ we have,

$$\frac{1}{2} (y')^2 \leq \frac{1}{2} p_0^2$$

or

$$|y'| \leq |p_0|.$$  

Since

$$y(t) = y_0 + \int_0^t y'(s) ds$$

it follows that

$$|y(t)| \leq |y_0| + |t|p_0$$

and so $|y(t)| < \infty$ for all $t$.

**Example 2.3.6**  
Consider the IVP

$$x'' + \alpha(x, x') x' + \beta(x) = u(t)$$

$$x(0) = x_0, \quad x'(0) = v_0$$

where $\alpha, \alpha_x, \beta, \beta'$ are continuous and $\alpha \geq 0, \beta(z) \geq 0$. We will show that all solutions are global.
First, it is a straightforward matter to verify that the IVP has a local solution for any initial data. If we multiply the differential equation by the solution, say $\xi(t)$, then
\[
\frac{d}{dt}\left(\frac{1}{2}(\xi')^2 + \int_{0}^{\xi(t)} \beta(s) \, ds\right) = -\alpha(\xi, \xi')(\xi')^2 + u(t)\xi'(t)
\]
\[
\leq u\xi' \leq \frac{1}{2}(u^2 + (\xi')^2).
\]
Since $z\beta(z) \geq 0$,
\[
\int_{0}^{\xi} \beta \, ds \geq 0.
\]
Call
\[
F(t) = \frac{1}{2}(\xi')^2 + \int_{0}^{\xi(t)} \beta(s) \, ds.
\]
Then
\[
F(t) \geq \frac{1}{2}(\xi')^2,
\]
and from the above inequalities we see
\[
F'(t) \leq \frac{1}{2}((\xi')^2 + u^2) \leq F(t) + \frac{1}{2}u^2,
\]
or
\[
F'(t) - F(t) \leq \frac{1}{2}u^2.
\]
Thus
\[
\frac{d}{dt}(e^{-t}F) \leq \frac{1}{2}e^{-t}u^2
\]
or
\[
F(t) - F(0) \leq e^t \int_{0}^{t} e^{-s}u^2(s) \, ds.
\]
Thus we may write
\[
\frac{1}{2}(\xi')^2 \leq F(t) \leq G(t)
\]
or
\[
|\xi'(t)| \leq H(t)
\]
where $G(t)$, $H(t)$ are functions that are finite for all $t$. With this bound on the derivative we then get
\[
|\xi(t)| \leq |x_0| + \int_{0}^{t} |\xi'(s)| \, ds < \infty, \text{ for all } t.
\]

The preceding examples and Theorem(2.3.3) are special cases of the next result.
Theorem 2.3.4  Assume \( f : U \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \), \( U \) open and \( f(t,x) \) continuous and locally Lipschitz with respect to the second variable. Let \( \xi(t) \) be the solution of the IVP

\[
x' = f(t,x) \\
x(t_0) = x_0
\]

and \((m_1,m_2)\) its maximal interval of existence. If \( m_2 < \infty \) and \( E \) is any compact subset of \( U \), then there exists an \( \epsilon > 0 \) such that \((t,\xi(t))\) is not in \( E \) if \( t > (m_2 - \epsilon) \) (and similarly for \( m_1 \)).

**Proof.** Consider the closed set \( U^c = \mathbb{R}^{n+1} - U \) and let \( d(E,U^c) = \rho > 0 \). Now pick a closed set \( E^* \subset U \) such that \( E \subset E^* \) and \( d(E,E^*) < \rho/2 \).

We will assume that \((t,\xi(t))\) \( \in E \) for all \( t \in (m_1,m_2) \) and obtain a contradiction. To this end, choose \( M \) such that \( |f(x,t)| \leq M \) for all \((t,x)\) \( \in E^* \) and select \( r < \rho/2 \). Pick any \((\tilde{t},\tilde{x})\) \( \in E \) and let

\[
K_r = \overline{B}_r(\tilde{x}).
\]

Note that if \((t,x) \in K_r\), \( \max\{|t-\tilde{t}|,|x-\tilde{x}|\} \leq r < \rho/2 \) and \( K_r \subset E^* \). The IVP has a unique solution that exists on an interval \( |t-\tilde{t}| < \epsilon \) where \( \epsilon < r, \epsilon M < r, \epsilon L < 1 \) and \( L \) is a Lipshitz constant on the set \( E^* \). Moreover, the same \( M \) and \( L \) will work for any \((\tilde{t},\tilde{x})\) since \( K_r \subset E^* \). Now select \( \hat{t} \in (m_2 - \epsilon, m_2) \). Then \((\hat{t},\xi(\hat{t}))\) \( \in E \) so the IVP

\[
x' = f(t,x) \\
x(\hat{t}) = \xi(\hat{t})
\]

has a unique solution \( \psi(t) \) that exists on \( |t-\hat{t}| \leq \epsilon \). Then

\[
\xi(t) = \begin{cases} 
\xi(t), & m_1 < t < \hat{t} \\
\psi(t), & \hat{t} \leq t < \hat{t} + \epsilon
\end{cases}
\]

is a continuation of \( \xi(t) \) defined on \((m_1,\hat{t} + \epsilon)\). But

\[
\hat{t} + \epsilon > m_2 - \epsilon + \epsilon > m_2
\]

contradicting the maximality of \((m_2,m_2)\).

## 2.4 Dependence on Data

In an initial value problem

\[
x' = f(t,x) \\
x(t_0) = x_0
\]
one might regard \( t_0, x_0 \) and \( f(t, x) \) as measured values or inputs in the formulations of a physical model. Consequently it is important to know if small errors or changes in this data would result in small changes in the solutions of IVP. That is, does the solution depend continuously on \((t_0, x_0)\) and \(f(t, x)\) in some sense.

Denote the solution the IVP by \( \xi(t, t_0, x_0) \) where

\[ \xi(t_0, t_0, x_0) = x_0. \]

We will show that under reasonable assumptions on \( f \), \( \xi \) is continuous in the variables \( t_0, x_0 \) and small changes in \( f \) result in small changes in \( \xi \). The following theorem is an indispensable result in the study of differential equations and is central to our results of this section.

**Theorem 2.4.1** [Gronwall’s Inequality] Let \( f_1(t), f_2(t), p(t) \) be continuous on \([a, b]\) and \( p \geq 0 \). If

\[ f_1(t) \leq f_2(t) + \int_a^t p(s)f_1(s) \, ds, \quad t \in [a, b], \]

then

\[ f_1(t) \leq f_2(t) + \int_a^t p(s)f_2(s) \exp\left[ \int_a^t p(u) \, du \right] \, ds. \]

**Proof.** Define

\[ g(t) = \int_a^t p(s)f_1(s) \, ds, \]

so

\[ g'(t) = p(t)f_1(t) \leq p(t)(f_2(t) + \int_a^t p(s)f_1(s) \, ds). \]

We then get

\[ g'(t) - p(t)g(t) \leq p(t)f_2(t), \]

\[ \frac{d}{dt}(g(t)e^{-\int_a^t p(u) \, du}) \leq p(t)f_2(t)e^{-\int_a^t p(u) \, du}, \]

\[ g(t)e^{-\int_a^t p(u) \, du} \leq \int_a^t p(s)f_2(s)e^{-\int_a^u p(u) \, du} \, ds \]

and

\[ g(t) \leq \int_a^t p(s)f_2(s)e^{\int_u^t p(u) \, du} \, ds. \]

Now \( f_1(t) \leq f_2(t) + g(t) \) and so the result follows.

There are some special cases of Gronwall’s inequality that should be noted.
(1) If \( p(x) = k \) and \( f_2(x) = \delta \) are constant, then Gronwall gives
\[
f_1(x) \leq \delta e^{k(x-a)}
\]

(2) If
\[
f_1(x) \leq k \int_a^x f_1(t) \, dt, \quad k \geq 0
\]
then \( f_1(x) \equiv 0 \).

(3) Suppose \( |z'(x)| \leq \mu |z(x)| \) for \( a \leq x \leq b \) and \( z(a) = 0 \), then
\[
\left| \int_a^x z'(t) \, dt \right| \leq \int_a^x |z'(t)| \, dt \leq \mu \int_a^x |z(t)| \, dt
\]
and so
\[
|z(x)| \leq \mu \int_a^x |z(t)| \, dt.
\]
It follows by (2), that \( |z(x)| \equiv 0 \).

**Theorem 2.4.2** Suppose \( \xi(t), \psi(t) \) satisfy
\[
y' = f(t, y)
y(t_0) = y_0
z' = g(t, z)
z(t_0) = z_0
\]
where \( f, g : U \subseteq \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n \), are continuous and locally Lipschitz with respect to the second variable with Lipschitz constant \( K \). If
\[
|f(t, u) - g(t, u)| \leq \epsilon, \ (t, u) \in U,
\]
then
\[
|\xi(t) - \psi(t)| \leq |y_0 - z_0|e^{K|t-t_0|} + \frac{\epsilon}{K}(e^{K|t-t_0|} - 1).
\]

**Proof.** First assume \( t \geq t_0 \). Then
\[
\xi(t) - \psi(t) = y_0 - z_0 + \int_{t_0}^t f(s, \xi(s)) - g(s, \psi(s)) \, ds
= y_0 - z_0 + \int_{t_0}^t [f(s, \xi(s)) - f(s, \psi(s))] \, ds.
\]
Thus
\[ |\xi(t) - \psi(t)| \leq |y_0 - z_0| + \epsilon(t - t_0) + K \int_{t_0}^{t} |\xi(s) - \psi(s)| \, ds. \]

Now apply Gronwall with
\[ f_1 = |\xi - \psi|, \quad f_2 = |y_0 - z_0| + \epsilon(t - t_0), \quad p = k. \]

Then
\begin{align*}
|\xi(t) - \psi(t)| &\leq \epsilon(t - t_0) + |y_0 - z_0| + K \int_{t_0}^{t} (|s - t_0| + |y_0 - z_0|) e^{K(t-s)} \, ds \\
&= \epsilon(t - t_0) + |y_0 - z_0| + K \left\{ \left( |s - t_0| + |y_0 - z_0| \right) \frac{e^{K(t-t_0)}}{-K} \right\} |t|_0 + \epsilon \int_{t_0}^{t} e^{K(t-s)} ds \\
&= \epsilon(t - t_0) + |y_0 - z_0| + K \left\{ \left( |s - t_0| + |y_0 - z_0| \right) \frac{1}{K} e^{K(t-t_0)} \right\} + \epsilon \left( \frac{e^{K(t-t_0)}}{-K} \right) |t|_0 \\
&= |y_0 - z_0| e^{K(t-t_0)} + \frac{\epsilon}{K} (e^{K(t-t_0)} - 1).
\end{align*}

If \( t < t_0 \), a similar argument gives
\[ |\xi(t) - \psi(t)| \leq |y_0 - t_0| e^{k(t_0-t)} + \frac{\epsilon}{K} (e^{k(t_0-t)} - 1) \]
and the result follows.

**Example 2.4.1** Consider the initial value problems,
\begin{align*}
(1) \quad & \begin{cases} 
    y' = f(t, y) = 1 + t^2 + y^2, & \text{Ricatti's Equation} \\
    y(0) = y_0 
\end{cases} \\
(2) \quad & \begin{cases} 
    z' = g(t, z) = 1 + z^2 \\
    z(0) = y_0 
\end{cases}
\end{align*}

Of course problem (2) is easily solved. If we were to approximate the solution to (2) by that of (1) on the set
\[ U = \{(t, u) ||t| < 1/2, \ |u| < 1\}, \]
we would like to estimate the error. In the notation of Theorem(2.4.2)
\[ |f(t, u) - g(t, u)| = |t^2| < \frac{1}{4} = \epsilon \]
Also
\[ \left| \frac{\partial f}{\partial u} \right| = |2u| \leq 2, \quad \left| \frac{\partial g}{\partial u} \right| = |2u| \leq 2 \]
and so we can take the common Lipschitz constant to be \( K = 2 \). Then
\[
|y(t) - z(t)| \leq \frac{\epsilon}{K} (e^{K|t-t_0|} - 1) \\
\leq \frac{1}{2} (e^{2(1)} - 1) \approx 0.2.
\]
If, however, we were to restrict \( |t| < 1/4 \) then we get a much better approximation,
\[
|y(t) - z(t)| \leq \frac{1}{32} (1.6487 - 1) \approx 0.0203
\]
Exercises for Chapter 2

1. A solution $y = \phi(x)$ to

$$y'' + \sin(x)y' + (1 + \cos(x))y = 0$$

is tangent to the x-axis at $x = \pi$. Find $\phi(x)$.

2. Show that the initial value problem

$$y' = \frac{1}{1 + y^2}, \quad y(0) = 1$$

has a unique solution that exists on the whole line.

3. Consider the initial value problem

$$y''(x) + F'(y) = 0, \quad y(x_0) = y_0, \quad y'(x_0) = v_0$$

(a) If $F \in C^2(\mathbb{R})$, carefully explain why the Fundamental Existence and Uniqueness theorem guarantees that this initial value problem has a unique solution for any point $(x_0, y_0) \in \mathbb{R}^2$.

(b) Suppose that $F(u) > 0, u \in \mathbb{R}$. Prove that the solution to the initial value problem exists for all $x \in \mathbb{R}$.

4. Consider the equation

$$y'(x) = \frac{xy}{1 + y^2} + \sin(x).$$

(a) Explain why for each $(x_0, y_0) \in \mathbb{R}^2$ there is a solution of the differential equation that satisfies $y(x_0) = y_0$ that is defined in some neighborhood of $x_0$.

(b) Show that any solution of the differential equation satisfies

$$|y(x)| \leq k_1 e^{k_2 x^2}$$

for constants $k_1, k_2$.

(c) Prove that each solution of the differential equation can be extended to all of $\mathbb{R}$.

5. Consider

$$y'' + q(x)y = 0$$

$$y(x_0) = y_0, \quad y'(x_0) = v_0$$

where $q \in C[a, b], x_0 \in [a, b]$.

(a) Carefully explain why this problem has a unique solution.

(b) Show that if a solution has a zero in $[a, b]$ it must be simple.
6. Consider the equation
\[ y'' + (1 + ap(x))y = 0 \]
where \( a \) is a nonnegative constant and \( p(x) \in C(\mathbb{R}), |p(x)| \leq 1 \). Let \( D \) be the domain \( D = \{(x,y)| 0 \leq x \leq \rho, \ 0 \leq y \leq 1\} \) and let \( y = \phi(x) \) denote the solution of the initial value problem
\[ y'' + (1 + ap(x))y = 0, \quad y(0) = 0, \quad y'(0) = 1. \]
Suppose we approximate the solution of the initial value problem by \( \sin(x) \) on the domain \( D \). Estimate \( \|\phi(x) - \sin(x)\| \) for \( 0 \leq x \leq \rho \).

7. Estimate the error in using the approximate solution \( y(x) = e^{-x^3/6} \)
\[ 0 \leq x \leq 1/2 \] for the initial value problem
\[ y''(x) + xy(x) = 0 \]
\[ y(0) = 1, \quad y'(0) = 0 \]