ELASTIC WAVES IN A THREE-DIMENSIONAL HALF SPACE: THE LAMB PROBLEM

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Abstract
The problem of the title is to find time-harmonic solutions of the elasticity equations in a three dimensional half space with boundary free of normal

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stresses occasioned by a time-harmonic source located in the half space. This is here accomplished by means of the limiting absorption principle. The asymptotic behavior for large spatial distances of the solutions so obtained is determined, and uniqueness classes containing these solutions are determined. It is further shown in what sense these solutions are approximations for large times to the actual time-dependent solution.

1 Introduction

The Lamb problem is to find time-harmonic solutions of the elasticity equations in a medium filling \( \mathbb{R}_+^3 = \{ x \in \mathbb{R}^3 : x_3 > 0 \} \) with boundary \( \{ x_3 = 0 \} \) free of normal stresses and a time-harmonic source located in \( \mathbb{R}_+^3 \) (see, e.g., [1, 2]). In the present work we construct such solutions via the principle of limiting absorption, determine their asymptotic behavior as \( |x| \to \infty \), define uniqueness classes containing them, and show that these unique solutions are the limits as \( t \to \infty \) of the actual time-dependent solution (multiplied by the time-harmonic exponential with argument of sign opposite that of the source) – the principle of limiting amplitude. Our reason for investigating this particular problem is that it is the simplest problem with infinite boundary for the elasticity equations which admits surface waves. The system of elasticity equations admits two positive propagation speeds \( c_s < c_p \) of pressure (P) and shear (S) waves, and with the
present boundary condition a P wave, for example, incident on \( \{x_3 = 0\} \) gives rise to a reflected P wave as well as an S wave and a surface (or Rayleigh (R)) wave created on \( \{x_3 = 0\} \). A similar statement holds for an incident S wave. This is a more general situation than cases for Maxwell’s equations admitting surface waves [7, 16] and provides additional information as to what can be expected for more general systems as well as providing a reference problem for more general domains.

The principle of limiting absorption consists roughly in assuming that the frequency has a small imaginary part and then, with an eventual solution in hand, letting this imaginary part go to zero (i.e., in considering the limit on the spectrum of the resolvent applied to the spatial part of the source). There are three basic problems involved here. The first is to establish the principle of limiting absorption itself, i.e., to show in a rigorous manner that a steady-state solution can actually be constructed in this fashion. The second problem is to find a class of functions in which the solution so constructed is unique (a “radiation condition”). While in problems in exterior domains or with bounded perturbations of the coefficients uniqueness classes are essentially dictated by the asymptotic behavior of the free-space Green functions (see, e.g., [17]), in problems with infinite boundary admitting surface waves, such as the present problem, the asymptotic behavior as \(|x| \to \infty \) of these latter must also be
considered in defining uniqueness classes (see [16]). Finally, since, strictly speaking, steady-state solutions are physically meaningless (they fail to have finite energy), a third problem is to determine in what sense they are approximations for large times to the actual time-dependent solutions (the principle of limiting amplitude). We proceed to describe the steps taken to resolve these problems in the case of the Lamb problem and to outline the results obtained.

In §2 the system of equations for nonstatic solutions, which is second order in both the time and spatial variables, is derived from the elasticity equations as a first-order, symmetric, hyperbolic system, and its relation to the system, first order in time and second order in the spatial variables, is established. In §3 the selfadjoint operator $M$ in $L_2(\mathbb{R}^3_+; C^3)$ engendered by the elasticity operator $M(D)$ in $\mathbb{R}^3_+$ with the condition that \( \{x_3 = 0\} \) be free of normal stresses is constructed and given a form suitable for determining the generalized eigenfunctions. These are of three types: 1) $\Sigma_R(x, \xi), \xi \in \mathbb{R}^2$, the surface or Rayleigh mode; 2) $\Psi_p(x, \eta), \eta \in \mathbb{R}^3$, consisting of an incident and reflected $P$ (pressure) mode plus an SVP mode – an SV (vertical shear) mode created at the boundary by the incident $P$ mode; 3) $\Psi_{sh}(x, \eta) + \Psi_{sv}(x, \eta)$, where the first consists of an incident and reflected $SH$ (horizontal shear) mode, while the second consists of an incident and reflected SV mode plus a PSV mode – a $P$ mode created at the boundary by
the incident SV mode. We remark that the PSV mode decays exponentially away from the boundary for \( \eta \) outside the cones 
\[
C_\pm = \{ \eta = |\eta|s = |\eta|(s', s_3), s \in S^2, |s'| = n, s_3 = \pm \sqrt{(1 - n^2)} \}
\]
where \( 1 > n = c_s/c_p \) is the ratio of the shear and pressure propagation speeds of the medium filling \( \mathbb{R}_+^3 \). The solution of the time-dependent equations is then represented as a superposition of these modes. Section 5 contains the formulation of the steady-state problem and a representation of the resolvent of \( M \) in terms of generalized eigenfunctions. (It may seem somewhat silly to construct the resolvent, then obtain generalized eigenfunctions, and then again represent the resolvent. The point is that the first construction contains the spectral parameter in many untoward places, while in the second it is contained only in denominators.) In §§6-8 the principle of limiting absorption is established, and the asymptotics of the various components of the steady-state solution as \(|x| \to \infty\) are determined. In §6 a very simple expression is obtained for the leading term of the asymptotics of the steady-state Rayleigh wave: in \( 0(|x|^{-1/2}) \) it (multiplied by the time exponential) is an outgoing or incoming cylindrical wave, decaying exponentially in the \( x_3 \) direction. In §7 an asymptotic expression is obtained for the component of the steady-state solution derived from superposition of the \( \Psi_{sh}(x, \eta) \) and \( \Psi_{sv}(x, \eta) \) modes: in \( 0(|x|^{-1}) \) the first is simply an outgoing or incoming \( SH \) wave (the reflected wave is manifest only in the coefficient), while the
SV part consists of an incoming or outgoing $P$ wave and $SV$ wave, uniformly with respect to direction. In §8 an expression is derived for the component of the steady-state solution derived from superposition of the $\Psi_p(x, \eta)$ modes: this consists of a $P$ wave in $0(|x|^{-1})$ plus an $SV$ wave present in $0(|x|^{-1})$ only in the cone $\tilde{C}_+ = \{|x|\omega : \omega \in S^2, |\omega'| < n\}$; outside this cone it decays like $0(|x|^{-1-\kappa})$, $\kappa \in (1/2, 1)$. Thus, although the $SVP$ modes evidence no exceptional behavior on the cones $C_{\pm}$ above, the $SVP$ steady-state wave exists in $0(|x|^{-1})$ only in the cone $\tilde{C}_+$, while the $PSV$ wave, for which modes from which it is formed decay exponentially in $\eta$-space outside the cones $C_{\pm}$, reflects no evidence of this asymptotically. All this is simply a consequence of the location of the critical points of the phases of the $PSV$ and $SVP$ modes. Nevertheless, in view of the usual sort of plane-wave-chasing analysis done in many applied works, it should be emphasized that the behavior of the $\Psi_p$ and $\Psi_s$ modes from which the steady-state solution is formed gives a totally inverted picture of the asymptotic behavior of the steady-state solution itself. It is an interesting fact that neither the $SVP$ wave nor $PSV$ wave can be observed on the boundary in $0(|x|^{-1})$: in the first case this happens because the wave vanishes in this order outside the cone $\tilde{C}_+$, while in the second case it happens because the reflection coefficient vanishes on the boundary. In §9 we formulate the asymptotics of the components of the steady-state solution obtained in §§6-8 as
Theorem 9.1, define eventual uniqueness classes containing
them, and then in Theorem 9.2 prove their uniqueness in
these classes. In §10 we establish that the unique steady-
state solutions of §9 are indeed approximations to the actual
solutions of §4. The appendix contains the proof of a tech-
nical result stated in §6 which is needed for the uniqueness
theorem.

As regards related literature, the Lamb problem in vari-
ous formulations has been considered in many works (see,
e.g., [1, 2] and the bibliographies there). For layered me-
dia with infinite interfaces of various shapes the limiting-
absorption principle was established for the Helmholtz equa-
tion by Eidus [3], but the abstract approach adopted there,
while relatively “clean” mathematically, apparently does not
afford enough information about the solution so obtained to
be able to define uniqueness classes. For Maxwell’s equa-
tions with dissipative boundary conditions essentially the
same problem as considered here was treated in [16]. Fi-
ally, we mention that in the formulation of this problem or
analogous problems with infinite boundary it is often stated
that the solution should satisfy a “radiation condition” at
infinity (see, e.g., [1, 2, 9]). However, to our knowledge the
said condition has never been explicitly formulated. For the
Helmholtz equations and a restricted class of domains with
infinite boundaries or interfaces a Sommerfeld-type radiation
condition for the Dirichlet problem is established in [22]. In
the present case and in general there is no such radiation condition: the asymptotic properties of the steady-state solutions constructed by limiting absorption are used to define uniqueness classes containing them (which is really what Sommerfeld-type conditions do implicitly in simple cases).

2 The Equations of Elasticity and the Associated Second-Order Systems

In this section we present the equations of elasticity as a first-order, symmetric, hyperbolic system and derive an associated second-order system for nonstatic solutions. This system gives rise to a system in energy space which is first order in time and second order in the spatial variables. The correspondence between these two systems has general character and is established for abstract systems. The correspondence of the resolvents and spectral families of the abstract systems is also established. The results will be applied to the concrete system considered in §3.

We write the equations of elasticity with an initial-value or initial-boundary-value problem in mind; source terms will be dealt with in §10. Everywhere below \( tM \) denotes the transpose of a matrix \( M \), \( ^{t}M \) denotes the conjugate transpose, and \( M^{*} \) denotes the adjoint of a matrix or operator. In terms of the stress tensor \( u^{1} = t(\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{13}, \sigma_{23}) \)
and the time derivative $u^2 = \iota(v_1, v_2, v_3) \equiv \iota(\partial_t w_1, \partial_t w_2,\partial_t w_3)$ of the elastic displacement $w = \iota(w_1, w_2, w_3)$ with $u = \iota(u_1, u^2)$ the equations of elasticity of a homogeneous, isotropic medium occupying a region of three-dimensional space can be written as the first-order hyperbolic system
[15] ($D_j = -i\partial_j, j = 1, 2, 3$),

$$-i\partial_t u = E^{-1}A(D)u,$$

$$A(D) = \begin{bmatrix} 0_{6\times6} & A_{6\times3} \\ tA_{3\times6} & 0_{3\times3} \end{bmatrix},$$

$$tA(D) = \begin{bmatrix} D_1 & 0 & 0 & D_2 & D_3 \\ 0 & D_2 & 0 & D_1 & 0 \\ 0 & 0 & D_3 & 0 & D_1 \end{bmatrix},$$

$$E^{-1} = \begin{bmatrix} E_{0}^{-1}_{6\times6} & 0_{6\times3} \\ 0_{3\times6} & I_3 \end{bmatrix},$$

$$E_{0}^{-1} = \begin{bmatrix} e_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & \mu I_3 \end{bmatrix},$$

$$e = \begin{bmatrix} c_p^2 & \lambda_2 & \lambda \\ \lambda & c_p^2 & \lambda_2 \\ \lambda & \lambda & c_p^2 \end{bmatrix}.$$  

Here $\lambda, \mu$ are the Lamé constants in terms of which the
propagation speeds of $S$ and $P$ waves, $c_s$ and $c_p$, are given by $c_s^2 = \mu$ and $c_p^2 = \lambda + 2\mu$ respectively; $-i\partial_t u^1 = E_0^{-1} \mathcal{A}(D) u^2$ is the time-derivative of Hooke’s law $\sigma_{ij} = \lambda \delta_{ij} \sum_1^3 e_{kk} + 2\mu e_{ij}$, $2e_{ij} = \partial_i w_j + \partial_j w_i, \, i, j = 1, 2, 3$, and in terms of the displacement $w$ the second equation $-i\partial_t u^2 = t^t \mathcal{A}(D) u^1$ can be written

$$\partial_t^2 w(x, t) = -t^t \mathcal{A}(D) E_0^{-1} \mathcal{A}(D) w(x, t) + ij \mathcal{A}(D) u^1_0(x), \quad (2.2)$$

where $u^1_0$ is the static (time-independent) component of $u(x, t)$. Since in this paper we shall be interested only in nonstatic solutions, we suppose that $u^1_0(x) = 0$, so that (2.2) can be written as the $3 \times 1$ system

$$\partial_t^2 w = -t^t \mathcal{A}(D) E_0^{-1} \mathcal{A}(D) w = \text{Div } \sigma(x, t)$$

$$(2.3)$$

where the divergence of the stress tensor is $\text{[Div } \sigma_j] = \sum_1^3 \partial_k \sigma_{jk}$, $j = 1, 2, 3$, and

$$\text{[grad div}_i j] = \partial_i \partial_j = \text{[grad div}_i j], \quad i, j = 1, 2, 3, \quad$$

$$\text{rot } = \begin{bmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{bmatrix}.$$
Given \( w \), the stress can be recovered as \( \sigma = iE_0^{-1}A(D)w \).

We shall thus be concerned with the elasticity equations in the form (2.3):

\[
\partial_t^2 w(x, t) + M(D)w(x, t) = 0,
\]

\[
M(D) = \, ^tA(D)E_0^{-1}A(D), \quad (2.4)
\]

\[
= -c_p^2 \partial \otimes \partial + c_s^2 \text{rot rot}.
\]

For the symbol \( M(\eta) = \, ^tA(\eta)E_0^{-1}A(\eta) \), \( 0 \neq \eta = |\eta|\omega \in \mathbb{R}^3, \omega \in S^2 \), we compute

\[
\det[M(\eta) - zI] = (z - c_p^2|\eta|^2)(z - c_s^2|\eta|^2)^2, \quad z \in \mathbb{C},
\]

\[
M(\eta) = c_p^2|\eta|^2P(\omega) + c_s^2|\eta|^2S(\omega),
\]

\[
I = P(\omega) + S(\omega),
\]

\[
P(\omega) = \omega \otimes \omega = \omega \omega = P^2(\omega),
\]

\[
S(\omega) = -\omega \wedge \omega \wedge = S^2(\omega),
\]

\[
= v(\omega) \otimes v(\omega) + h(\omega) \otimes h(\omega),
\]

\[
v(\omega) = |\omega'|^{-1} \, ^t(\omega_1\omega_3, \omega_2\omega_3, -|\omega'|^2),
\]

\[
h(\omega) = |\omega'|^{-1} \, ^t(\omega_2, -\omega_1, 0),
\]

(2.5)
ω∧ = \begin{bmatrix} 0 & -ω_3 & ω_2 \\ ω_3 & 0 & -ω_1 \\ -ω_2 & ω_1 & 0 \end{bmatrix},

(ω∧)^2 = ω∧ω∧,

|ω'|^2 = ω_1^2 + ω_2^2.

Here v(ω) ⊗ v(ω) and h(ω) ⊗ h(ω) are, respectively, the SV and SH components of S(ω), i.e., 0 = ω · h(ω) = ω · v(ω) = h(ω) · v(ω) and ω ∧ h(ω) = v(ω), ω ∧ v(ω) = −h(ω).

As is known [21], the Fourier transform

\[ \Phi_n f(η) \equiv \hat{f}(η) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \exp(-ixη)f(x)dx \]

is an automorphism of the Schwartz space \( S(\mathbb{R}^n) \) of smooth, rapidly decreasing functions which extends by continuity to an automorphism of \( L_2(\mathbb{R}^n) \) and by duality to an automorphism of \( S'(\mathbb{R}^n) \). The symbol \( Φ \) with no index means \( Φ_3 \). In terms of the Fourier transform the fundamental solutions \( g_{p,s} \) of the Helmholtz equation for Im \( \sqrt{z} > 0 \), \( z \in \mathbb{C} \setminus \mathbb{R}_+ \),

\[ (\Delta + c_{p,s}^{-2}z)g_{p,s} = −δ, \]

have Fourier transforms

\[ \hat{g}_{p,s}(η; z) = (2\pi)^{-3/2}c_{p,s}^2(c_{p,s}^2|η|^2 - z)^{-1}, \]
and for the Fourier transform of $I(x; z)$,

$$[M(D) - zI_3]I(x; z) = \delta I_3,$$

we have

$$(2\pi)^{3/2} \hat{I}(\eta; z) = [M(\eta) - zI_3]^{-1}$$

$$= (c_p^2|\eta|^2 - z)^{-1}P(\omega) + (c_s^2|\eta|^2 - z)^{-1}S(\omega)$$

$$= z^{-1}[c_p^2|\eta|^2(c_p^2|\eta|^2 - z)^{-1}P(\omega) +
  c_s^2|\eta|^2(c_s^2|\eta|^2 - z)^{-1}S(\omega) - I_3]$$

$$= (2\pi)^{3/2}z^{-1}||\eta|^2P(\omega)\hat{g}_p(\eta, z) +$$

$$+||\eta|^2S(\omega)\hat{g}_s(\eta, z) - (2\pi)^{-3/2}I_3$$

so that, taking the inverse Fourier transform in $S'$,

$$-I(x; z) = z^{-1}[\partial \otimes \partial g_p(x; z) - \text{rot rot} g_s(x; z) + \delta I_3].$$

Computed explicitly in $S'$,

$$-zI(x; z) = -k_p^2 g_p(x; z) + k_s^2 g_s(x; z) + ik_p g_p(x; z)|x|^{-1}(I - 3\omega \otimes \omega)$$

$$+ik_s g_s(x; z)|x|^{-1}(2I + 3\omega \wedge \omega)$$

$$= g_p(x; z)|x|^{-2}(3\omega \otimes \omega - I) - g_s(x; z)|x|^{-2}(2I + 3\omega \wedge \omega)$$

where the last two terms applied to $\phi \in S$ are to be understood in the sense of singular integrals. (The integrals over $S^2$ of both $I - 3\omega \otimes \omega$ and $2I + 3\omega \wedge \omega$ are equal to zero.)
Remark 2.1 It is a straightforward computation to verify that $I(x; z)$ can be written in the equivalent form \[10\]

$$I(x; z) = -zI_3g(x, z) + c_s^2(-\partial \otimes \partial)g + c_p^2 \text{rot rot } g,$$

$$g = [z(c_s^2 - c_p^2)]^{-1}g_p + [z(c_p^2 - c_s^2)]^{-1}g_s$$

$$= [z(c_s^2 - c_p^2)]^{-1}(g_p - g_s).$$

It is interesting to note that the fundamental solution of the second-order elliptic operator $M$ has a singularity at zero of $0(|x|^{-3})$, while first-order elliptic operators can have a singularity only like $0(|x|^{-2})$ [18].

Defining the selfadjoint operator $M$ in $L_2(\mathbb{R}^3; \mathbb{C}^3)$ by $\mathcal{D}(M) = \{f \in L_2(\mathbb{R}^3; \mathbb{C}^3) : Mf \in L_2(\mathbb{R}^3; \mathbb{C}^3)\}$, its resolvent is given by $(z \in \mathbb{C} \setminus \mathbb{R}_+)$

$$[M - zI]^{-1}f(x) = \Phi^*[M - z]^{-1}\hat{f}(x) \equiv I(z)f(x) = \int I(x - y; z)f(y)dy.$$

The nonnegative, selfadjoint operator $M$ in $L_2(\mathbb{R}^3; \mathbb{C}^3)$ generates a unitary group which solves the Schrödinger problem for $M$, while we are interested in the Cauchy problem for (2.4) with $w(x, 0) = w_0(x)$, $w_t(x, 0) = v_0(x)$. The solution of this problem can be expressed in terms of the square root of $M$. To see this it is convenient to go over to the energy
space where the corresponding group directly delivers the solution of the Cauchy problem. The following result is “well known”, but we know of no reference and therefore present the simple proof. Here \( M \geq 0 \) is any nonnegative, selfadjoint operator in a Hilbert space \( \mathcal{H} \) for which zero is not in its point spectrum \( \sigma_p(M) \).

**Theorem 2.1** Let \( M \geq 0 \) be a selfadjoint operator in a Hilbert space \( \mathcal{H} \) with domain \( \mathcal{D}(M) \) and with \( 0 \notin \sigma_p(M) \). Let \( \mathcal{H}_M \) be the completion of \( \mathcal{D}(M) \times \mathcal{H} \) in the norm \( [ |M^{1/2}f_1|^2 + |f_2|^2]^{1/2} \). Let
\[
\mathcal{M} = -i \begin{bmatrix} 0 & I \\ -M & 0 \end{bmatrix}
\]
with \( \mathcal{D}(\mathcal{M}) = \mathcal{D}(M) \times \mathcal{D}(M^{1/2}) \subset \mathcal{H}_M \). Then \( \mathcal{M} \) is selfadjoint in \( \mathcal{H}_M \).

**Proof.** Let \( f, g \in \mathcal{D}(\mathcal{M}) \). Since \( f_1 \in \mathcal{D}(M), f_2 \in \mathcal{D}(M^{1/2}) \),
\[
(f, \mathcal{M}g)_{\mathcal{H}} = (M^{1/2}f_1, -iM^{1/2}g_2)_{\mathcal{H}} + (f_2, iMg_1)_{\mathcal{H}}
\]
\[
= (iMf_1, g_2)_{\mathcal{H}} + (-iM^{1/2}f_2, M^{1/2}g_1)_{\mathcal{H}}
\]
\[
= (\mathcal{M}f, g)_{\mathcal{H}_M}.
\]
Hence, \( \mathcal{M} \) is symmetric, and so its defect indices are equal. Suppose now that \( f \in \mathcal{H}_M \) is orthogonal to the range of
\( \mathcal{M} - iI \), i.e., for all \( g \in \mathcal{D}(\mathcal{M}) \)

\[
0 = (f, (\mathcal{M} - iI)g)_{\mathcal{H}_M} = (M^{1/2}f_1, -iM^{1/2}g_2 - iM^{1/2}g_1)_{\mathcal{H}} 
\]
\[+ (f_2, iMg_1 - ig_2)_{\mathcal{H}}. \tag{2.12}\]

We first take \( g_1 = 0 : 0 = (M^{1/2}f_1, M^{1/2}g_2) + (f_2, g_2) \) for all \( g_2 \in \mathcal{D}(M^{1/2}) \) implies that

\[
M^{1/2}f_1 \in \mathcal{D}(M^{1/2} = M^{1/2}) \text{ and } M^{1/2}(M^{1/2}f_1) = -f_2. \tag{2.13}\]

We now take \( g_2 = 0: \) from (2.12), (2.13)

\[
(-f_2, g_1) = (M^{1/2}f_1, M^{1/2}g_1) = (f_2, Mg_1)
\]
for all \( g_1 \in \mathcal{D}(M) \) implies that \( f_2 \in \mathcal{D}(M) \) and \( Mf_2 = -f_2 \), so \( f_2 = 0 \), since \(-1 \not\in \sigma(M)\). Thus, from (2.13) \( M^{1/2}f_1 = 0 \), because \( 0 \not\in \sigma_p(M) \): suppose there exists \( g \in \mathcal{D}(M^{1/2}) \) such that \( M^{1/2}g = 0 \); then for all \( k \in \mathcal{D}(M) 0 = (M^{1/2}g, M^{1/2}k) = (g, Mk) \) which implies that \( g \in \mathcal{D}(M) \) and \( Mg = 0 \), so that \( g = 0 \). Thus, \( |f|_{\mathcal{H}_M} = 0 \), so \( f = 0 \) in \( \mathcal{H}_M \).

Hence, \( (\mathcal{M}^* + iI)f = 0 \) in \( \mathcal{H}_M \) implies that \( f = 0 \), and so the defect indices of \( \mathcal{M} \) are zero. The theorem is proved.

**Corollary 2.1** If \( z^2 \in \rho(M) \), the resolvent set of \( M \), then \( z \in \rho(\mathcal{M}) \), and in terms of the resolvent \( r(z^2) = (M - z^2I)^{-1} \) in \( \mathcal{H} \) the resolvent \( R(z) = (\mathcal{M} - zI)^{-1} \) in \( \mathcal{H}_M \) can be expressed on a dense set in \( \mathcal{H}_M \), e.g., \( \mathcal{D}(\mathcal{M}) \), as

\[
R(z) = \begin{bmatrix}
  zr(z^2) & -ir(z^2) \\
  iz^2r(z^2) + i & zr(z^2)
\end{bmatrix}. \tag{2.14}
\]
Proof. Let \( f \in \mathcal{D}(\mathcal{M}) \). Then

\[
\mathcal{D}(\mathcal{M}) \ni R(z)f = \begin{bmatrix}
zr(z^2)f_1 - ir(z^2)f_2 \\
nz^2r(z^2)f_1 + if_1 + rz(z^2)f_2
\end{bmatrix}.
\]

We must first show that \( |R(z)f|^2_{\mathcal{H}_M} \leq c(z)|f|^2_{\mathcal{H}_M} \). We suppose that \( f_2 = 0 \) and note that \( r(z^2)M^{1/2} \subset M^{1/2}r(z^2) \) [8, pp. 279, 285]; then

\[
|R(z)f|^2_{\mathcal{H}_M} = |M^{1/2}r(z^2)f_1|^2_{\mathcal{H}} + |z^2r(z^2)f_1 + f_1|^2_{\mathcal{H}}
\]

\[
= |z|^2r(z^2)M^{1/2}f_1|^2_{\mathcal{H}} + (Mr(z^2)f_1, z^2r(z^2)f_1)_{\mathcal{H}}
\]

\[
+ (r(z^2)f_1, f_1)_{\mathcal{H}}
\]

\[
\leq \left( |z|^2/|\text{Im } z^2| \right) |M^{1/2}f_1|^2_{\mathcal{H}} + |z^2| |M^{1/2}r(z^2)f_1|^2_{\mathcal{H}}
\]

\[
+ (r(z^2)f_1, f_1)_{\mathcal{H}}
\]

\[
\leq c(z)|M^{1/2}f_1|^2_{\mathcal{H}}.
\]

Now let \( f_1 = 0 \):

\[
|R(z)f|^2_{\mathcal{H}_M} = |M^{1/2}r(z^2)f_2|^2 + |z^2r(z^2)f_2|^2
\]

\[
= (r(z^2)f_2, Mr(z^2)f_2) + |z|^2|r(z^2)f_2|^2
\]

\[
= z^2|r(z^2)f_2|^2 + |z|^2|r(z^2)f_2|^2 + (r(z^2)f_2, f_2)
\]

\[
\leq c(z)|f_2|^2.
\]
Thus, \( R(z) \) is bounded on \( \mathcal{D}(\mathcal{M}) \), and it is straightforward to verify that for \( f \in \mathcal{D}(\mathcal{M}) \) \((\mathcal{M} - zI)R(z)f = f \). Hence, (2.14) is an expression for the resolvent on a dense set.

**Corollary 2.2** Suppose the spectrum of \( M \), \( \sigma(M) \), is continuous and \( \sigma(M) = [0, \infty) \). Then \( \sigma(\mathcal{M}) = \mathbb{R} \) is also continuous, and the spectral family for \( \mathcal{M} \) in \( \mathcal{H}_M \), \( E(d\lambda) \), is given in terms of the spectral family \( e(d\lambda^2) \) of \( M \) in \( \mathcal{H} \) by

\[
E(\delta) = 2^{-1} \int_\delta \begin{bmatrix}
  e(d\lambda^2) & -i\lambda^{-1}e(d\lambda^2) \\
  i\lambda e(d\lambda^2) & e(d\lambda^2)
\end{bmatrix} \text{sgn}(\lambda), \quad \delta \subset \mathbb{R},
\]

(2.15)

where if \( \delta = (\mu_1, \mu_2) \subset (-\infty, 0) \), this means that

\[
E(\mu_1, \mu_2) = 2^{-1} \int_{|\mu_1|}^{|\mu_2|} \begin{bmatrix}
  e(d\lambda^2) & i\lambda^{-1}e(d\lambda^2) \\
  -i\lambda e(d\lambda^2) & e(d\lambda^2)
\end{bmatrix}.
\]

**Proof.** The expression (2.15) is computed directly from (2.14) and Stone’s formula [8]

\[
E(\mu) = (2\pi i)^{-1} \lim_{\epsilon \to 0} \int_{-\infty}^{\mu} [R(\lambda + i\epsilon) - R(\lambda - i\epsilon)]d\lambda.
\]

Theorem 2.1 and its corollaries are stated for abstract operators. In the case of the operators \( M, \mathcal{M} \) considered below which are engendered by the elasticity operator \( M(D) \) plus the boundary condition equality (2.15), for example, makes it possible to compute the eigenfunction expansion
for $\mathcal{M}$ in the energy space directly from the eigenfunction expansion for $M$ in $\mathcal{H} = L_2(\mathbb{R}^3_+, \mathbb{C}^3)$ (see §4). Further for the unitary group $U(t) = \exp(i\mathcal{M}t)$ in $\mathcal{H}_M$ for $f \in \mathcal{H}_M$

$$U(t)f = \int_{-\infty}^{\infty} \exp(i\lambda t)E(d\lambda)f,$$  

(2.16)

so that for $f \in \mathcal{D}(\mathcal{M})$ from (2.15)

$$u(\cdot, t) = [U(t)f]_1$$

$$= \int_0^{\infty} [\cos(t\sqrt{\nu})f_1 + \nu^{-1/2}\sin(t\sqrt{\nu})f_2]e(d\nu)$$

$$= \cos(M^{1/2}t)f_1 + M^{-1/2}\sin(M^{1/2}t)f_2,$$

which is the usual expression for the solution of the Cauchy problem

$$u_{tt} + Mu = 0, u(0) = f_1, u_t(0) = f_2.$$  

(2.17)

(cf. [20, p. 15]).

We remark that in the case of a boundary value problem where the operator $E^{-1}A(D)$ plus a boundary condition engenders a selfadjoint operator its resolvent and spectral family (on the complement of the null space) can also be expressed in terms of those for $M$. We shall have no need for them in the present work.
3  The Elasticity Operator in $\mathbb{R}^3_+$ with Boundary Free of Normal Stresses

The operator $M(D) = \,^t\mathcal{A}(D)E_0^{-1}\mathcal{A}(D)$ of (2.4) and the boundary operator $(n = (0,0,1))$

$$B(D) \equiv \,^t\mathcal{A}(n)E_0^{-1}\mathcal{A}(D) = \begin{bmatrix} \mu D_3 & 0 & \mu D_1 \\ 0 & \mu D_3 & \mu D_2 \\ \lambda D_1 & \lambda D_2 & c_p^2 D_3 \end{bmatrix}$$

(3.1)

give rise to a selfadjoint operator in $\mathcal{H} = L_2(\mathbb{R}^3_+, \mathbb{C}^3)$. We recall from §2 that the components of the stress tensor are recovered from the displacement $w$ by $\sigma = iE_0^{-1}\mathcal{A}(D)w$. From (2.1) with $D \rightarrow n \, B(D)w(x',0) = 0$ then says that $\sigma_{i3}(x',0) = 0$, $i = 1, 2, 3$, i.e., the normal components of the stress are zero on the boundary.

Remark. In general, for any vector $n = (n^1,n^2,n^3)$ and displacement vector $w$ (sum on $i$)

$$\,^t\mathcal{A}(n)E_0^{-1}\mathcal{A}(D)w = \,^t(n^i\sigma_{i1},n^i\sigma_{i2},n^i\sigma_{i3}).$$

(3.2)

Definition 3.1 We denote by $\mathcal{D}(M)$ the set of compactly supported, smooth functions $f$ in $\mathbb{R}^3_+$ such that $B(D)f(x',0) = 0$. We define $M$ as the graph closure in $\mathcal{H} = L_2(\mathbb{R}^3_+, \mathbb{C}^3)$ of $M(D)$ on $\mathcal{D}(M)$ and denote its domain by $\mathcal{D}(M)$.

Theorem 3.1 The operator $M$ is a nonnegative, selfadjoint operator in $\mathcal{H}$.
It is clear by integration by parts on $\hat{\mathcal{D}}(M)$ that $M$ is symmetric and nonnegative. It is possible to show that its defect indices are zero by introducing the necessary Sobolev machinery as in [8, 9], but, rather than do this, we show that the equation $Mu + iIu = f$ has a solution for any $f \in \mathcal{H}$. This demonstrates that the operator $M$ has zero defect indices, and the operator $H(z), z \in \mathbb{C}\setminus\mathbb{R}_+$, delivering such solutions is its resolvent. We proceed to construct this operator explicitly.

Let $\phi \in \mathcal{S} = \mathcal{S}(\mathbb{R}^3, \mathbb{C}^3)$, and let $I$ and $\hat{I}$ be the distributions of (2.7), (2.8). We write $\mathbb{R}^3 \ni \eta = (\xi, \rho), \xi \in \mathbb{R}^2, \rho \in \mathbb{R}$, and denote by $\chi$ the characteristic function of $\{\eta = (\xi, \rho) : |\xi| < h, |\rho| < h\}$. Omitting the $z$-dependence of $I, \hat{I}$ to simplify notation, we then have

$$I(\phi) = \Phi^* \hat{I}(\phi) = \hat{I}(\Phi^* \phi) = (2\pi)^{-3/2} \int \hat{I}(\eta) \exp(i\eta x) \phi(x) dx d\eta$$

$$= \lim_{h \to \infty} I_h(\phi),$$

$$I_h(x) = (2\pi)^{3/2} \int \exp(i\eta x) \chi(\eta) \hat{I}(\eta) d\eta.$$  

For $\phi \in \mathcal{D} = C_0^\infty(\mathbb{R}^3, \mathbb{C}^3)$ by the continuity of differentiation and convolution in the dual space $\mathcal{D}'$

$$BI^* \phi(x) = \lim_{h \to \infty} BI^*_h \phi(x),$$

$$BI_h(x - y) = (2\pi)^{-3/2} \int \exp[i\eta(x - y)] \chi(\eta) B(\eta) \hat{I}(\eta) d\eta, \quad (3.3)$$
\[ \Phi_2 B I_h(\xi, 0, y) = (2\pi)^{-1/2} \chi_h(\xi) \exp(-iy') \int_{-h}^{h} \exp(-iy_3\rho) B(\eta) \bar{I}(\eta) d\rho. \]

The idea is now to evaluate the last integral explicitly in order to construct a function satisfying the boundary conditions. From (2.8) for \( \xi \neq 0 \) \( P(\omega) = |\eta|^{-2} \eta \otimes \eta \) and \( S(\omega) = |\eta|^{-2} (-\eta \wedge \eta \wedge) \) are analytic functions of \( \rho \) and so then also is \( \bar{I}(\eta; z) \). We extend \( \rho \) to \( \tau \in \mathbb{C} \), so that \( P(\xi, \tau) = (|\xi|^2 + \tau^2)^{-1}(\xi, \tau) \otimes (\xi, \tau) \), \( S(\xi, \tau) = (|\xi|^2 + \tau^2)^{-1} [-(-\xi, \tau) \wedge (\xi, \tau) \wedge] \) and define \( \tau_{p,s} = c_{p,s}^{-1}(z - c_{p,s}^2|\xi|^2)^{1/2} \) by \( \text{Im} \tau_{p,s} > 0 \) for \( z \in \mathbb{C} \setminus \mathbb{R}^+ \), noting that \( c_{p,s}^2|\xi|^2 + c_{p,s}^2 \tau^2 - z = c_{p,s}^2(\tau - \tau_{p,s}) \) \( (\tau + \tau_{p,s}) \). With \( y_3 > 0 \) we now choose \( h \) in (3.3) so large that the semicircle of radius \( h \) in \( \mathbb{C}_- \) encloses \( -\tau_{p,s} \). Then by the residue theorem

\[ \Phi_2 B I_h(\xi, 0, y; z) = \]

\[ = i(2\pi)^{-1} \chi_h(\xi) \exp(-iy') \exp(iy_3\tau_p)(2c_{p,s}^2\tau_p)^{-1} B(\xi, -\tau_p) \]

\[ P(\xi, -\tau_p) + \exp(iy_3\tau_s)(2c_{s,s}^2\tau_s)^{-1} B(\xi, -\tau_s) S(\xi, -\tau_s) \] + o_h(1),

\[ o_h(1) \leq \text{const} \int_0^\pi \exp(-y_3h \sin \theta) d\theta; y_3 > 0. \]

Hence,

\[ \Phi_2 B I^* \phi(\xi, 0) = \lim_{h \to \infty} \Phi_2 B I_h \phi(\xi, 0) \]

22
\[
\lim_{h \to \infty} \int \Phi_2 BI_h(\xi, 0, y; z) \phi(y) dy = \int \Phi_2 BI(\xi, 0, y; z) \phi(y) dy,
\]

\[
\Phi_2 BI(\xi, 0, y; z) = i(2\pi)^{-1} \exp(-iy^\prime \xi)((2c_p^2 \tau_p)^{-1} \exp(iy_3 \tau_p)B(\xi, -\tau_p)
\]

\[
P(\xi, -\tau_p) + (2c_s^2 \tau_s)^{-1} \exp(iy_3 \tau_s)B(\xi, -\tau_s)S(\xi, -\tau_s),
\]

\[
P(\xi, -\tau_p) = c_p^2 z^{-1}(\xi, -\tau_p) \otimes (\xi, -\tau_p), \quad (3.4)
\]

\[
S(\xi, -\tau_s) = c_s^2 z^{-1}[-(\xi, \tau_s) \wedge (\xi, -\tau_s)].
\]

We define

\[
P_\pm = t(\xi, \pm \tau_p),
\]

\[
V_\pm = |\xi|^{-1} t(\pm \xi_1 \tau_s, \pm \xi_2 \tau_s, -|\xi|^2) \quad (3.5)
\]

\[
H = |\xi|^{-1} t(c_s^{-1} \xi_2 \sqrt{z}, -c_s^{-1} \xi_1 \sqrt{z}, 0).
\]

Then

\[
c_s^{-2} z S(\xi, -\tau_s) = V_+ t V_- + H t H,
\]

\[
c_p^{-2} z P(\xi, -\tau_p) = P_- t P_-,
\]

\[
\Phi_2 BI(\xi, 0, y; z) = \alpha_p B(\xi, -\tau_p)P_- t P_- + \alpha_s B(\xi, -\tau_s)[V_+ t V_- + H t H], \quad (3.6)
\]

23
\[ \alpha_{p,s} = i(2\pi)^{-1} \exp(-iy'\xi)(2\tau_{p,s})^{-1} \exp(iy_3\tau_{p,s}). \]

We now construct an operator \( H(z) = I(z) - R(z) \) where \( I(z) \) is given in (2.11) and for \( f \in \mathcal{D}(\mathbb{R}_+^3) = C^\infty_0(\mathbb{R}_+^3; \mathbb{C}^3) \)

\[ R(z)f(x) = \int R(x, y; z) f(y) dy. \] (3.7)

\( R(z) \) is to be constructed so that \( (M - zI)R(z)f = 0 \) and \( BI(z)f(x', 0) = BR(z)f(x', 0) \). We seek \( R(x, y; z) \) in the form

\[ \Phi_2 R(\xi, x_3, y; z) = \alpha_p [r_{pp} \exp(i\tau_p x_3) P_+^i P_- + r_{sp} \exp(i\tau_s x_3) V_+^i P_-] \]

\[ + \alpha_s [r_{ss} \exp(i\tau_s x_3) V_+^i V_- + r_{ps} \exp(i\tau_p x_3) P_+^i V_-] \]

\[ + \alpha_s r_{hh} \exp(i\tau_s x_3) H^i H. \] (3.8)

Setting \( B(\pm \tau_{p,s}) = B(\xi, \pm \tau_{p,s}) \), we see from (3.6), (3.7) that

\[ \Phi_2 BI(\xi, 0, y; z) = \Phi_2 BR(\xi, 0, y; z) \]

if and only if

\[ B(-\tau_p) P_- = r_{pp} B(\tau_p) P_+ + r_{sp} B(\tau_s) V_+, \]

\[ B(-\tau_s) V_- = r_{ps} B(\tau_p) P_+ + r_{ss} B(\tau_s) V_+, \]

\[ B(-\tau_s) H = r_{hh} B(\tau_s) H. \]

Solving this system of equations, we obtain

\[ r_{pp} = r_{ss} = \tilde{\Delta}/\Delta, \]

24
\[ \Delta = (\tau_s^2 + |\xi|^2)^2 + 4|\xi|^2\tau_p\tau_s, \]
\[ \tilde{\Delta} = (\tau_s^2 + |\xi|^2)^2 - 4|\xi|^2\tau_p\tau_s, \quad (3.9) \]
\[ r_{sp} = -4|\xi|\tau_p(\tau_s^2 - |\xi|^2)/\Delta, \]
\[ r_{ps} = 4|\xi|\tau_s(\tau_s^2 - |\xi|^2)/\Delta. \]

We observe that
\[
0 = [M(D) - zI]R(x, y; z) = \Phi^*_2[M(\xi, D_3) - zI]\Phi_2R(\xi, x_3, y; z), \quad (3.10)
\]
since
\[
0 = [M(\xi, \tau_p) - zI]P_+ = [M(\xi, \tau_s) - zI]V_+ = [M(\xi, \tau_s) - zI]H.
\]
Further, for \( f \in \mathcal{D}(\mathbb{R}_+^3) \)
\[
H(z)f(x) = \int H(x, y; z)f(y)dy
= I(z)f(x) - R(z)f(x) \quad (3.11)
= \int I(x, y; z)f(y)dy - \int R(x, y; z)f(y)dy
\]
satisfies the boundary conditions by construction. This can easily be checked directly from (3.6), (3.8), (3.9).

**Theorem 3.2** For any \( f \in \mathcal{H} \) and \( z \in \mathbb{C}\setminus\mathbb{R}_+ \) \( H(z) \) is bounded in \( \mathcal{H} \), \( H(z)f \in \mathcal{D}(M) \), and \( (M - zI)H(z)f = f \). In particular, \( M \) is selfadjoint, and \( H(z) \) is its resolvent.
Proof. For \( f \in \mathcal{H} \) \(|I(z)f| \leq |\text{Im } z|^{-1}|f|\). From (3.6), (3.7), (3.8)
\[
\Phi_2 R(z)f(\xi, x_3) = \\
= i(2z\tau_p)^{-1}[r_{pp} \exp(i\tau_{p,x_3})P_+^tP_+ + r_{sp}\exp(i\tau_{s,x_3})V_+^tP_+]h_p \\
+ i(2z\tau_s)^{-1}[r_{ss}\exp(i\tau_{s,x_3})V_+^tV_+ + r_{ps}\exp(i\tau_{p,x_3})P_+^tV_-]h_s \\
- i(2z\tau_s)^{-1}\exp(i\tau_{s,x_3})H^tHh_s,
\]
\[
h_{p,s}(\xi, z) = \int \exp[i\tau_{p,s,y_3}]\Phi_2 f(\xi, y_3)dy_3
\]
\[
|h_{p,s}(\xi, z)|^2 \leq [2\text{Im } \tau_{p,s}(\xi, z)]^{-1} \int |\Phi_2 f(\xi, y_3)|^2dy_3.
\]

For large \(|\xi|\) from (3.5), (3.9) \(r_{pp}P_+^tP_+, r_{ss}V_+^tV_-\), etc. are at most \(0(|\xi|^2)\), and hence
\[
\int_0^\infty |\Phi_2 R(z)f(\xi, x_3)|^2dx_3 \leq c(z)(1+|\xi|)[|h_p(\xi, z)|^2+|h_s(\xi, z)|^2].
\]

Therefore,
\[
|R(z)f|^2 = |\Phi_2 R(z)f|^2 \leq c(z) \int R^2 (1 + |\xi|)[|h_p(\xi, z)|^2 + |h_s(\xi, z)|^2]d\xi \\
\leq c(z) \int_{R^2}^\infty \int_0^\infty |\Phi_2 f(\xi, y_3)|^2dy_3d\xi = c(z)|f|^2.
\]

Thus, \( H(z) \) is bounded. For \( f \in \mathcal{D}(\mathbb{R}^3) \) it is easy to see from (2.11), (3.7), (3.8) that \( H(z)f(x) \) is smooth and \( B(D)H(z)f(x', 0) = \)
Let $\phi$ be a smooth, scalar-valued function on $(0, \infty)$ such that $\phi(t) = 1$ for $0 < t < 1$ and $\phi(t) = 0$ for $t > 2$; let $\phi_N(x) = \phi(|x| - N)$. Then $\phi_N H(z) f \in D(M)$ and converges to $H(z) f$ in $\mathcal{H}$. Furthermore, the derivatives of $\phi_N$ are supported in $\{N - 1 \leq |x| \leq N + 1\}$ and are uniformly bounded with respect to $N$; hence, $M \phi_N H(z) f = \phi_N MH(z) f + o(1)$, so that $M \phi_N H(z) f \to MH(z) f$. Thus, $H(z) f \in D(M)$, and $(M - zI) H(z) f = f$. For any $f \in \mathcal{H}$ we approximate $f$ by functions in $D(\mathbb{R}^3_+)$ and use the fact that $H(z)$ is bounded and $M$ is closed (cf., e.g., [6]).

Because of the way $z$ occurs in $R(x, y; z)$, the present form of $H(x, y; z)$ is not convenient for establishing the limiting-absorption principle. For this an expression for $H(z)$ in terms of generalized eigenfunctions will be obtained in the next section. In order to be able to compute them we multiply $H(x, y; z)$ by the characteristic function $\chi_+ = \chi_+(y)$, $y \in \mathbb{R}^3_+$, and obtain an expression for the Fourier transform $\Phi_+ \chi_+ H(x, \eta; z)$.

For fixed $x \in \mathbb{R}^3_+$ let $\lambda_x \in D(\mathbb{R}^3_+)$, supp $\lambda_x \subset N_\delta(x) \subset \mathbb{R}^3_+$, $\lambda_x(y) = 1$ for $y \in N_{\delta/2}(x)$. For each $x \in \mathbb{R}^3_+$ and $\psi \in S(\mathbb{R}^3)$

$$\chi_+ (\cdot) H(x, \cdot; z)(\psi) = H(x, \cdot; z)(\lambda_x \psi) + \chi_+ (\cdot) H(x, \cdot; z)[(1 - \lambda_x) \psi]$$

Now $\lambda_x \psi \in D(\mathbb{R}^3_+)$, so $\lambda_x \chi_+(\cdot) H(x, \cdot; z) = \lambda_x H(x, \cdot; z) \in S'(\mathbb{R}^3)$; $[1 - \lambda_x(y)] H(x, y; z)$ is a smooth function, so $\chi_+ (\cdot) H(x, \cdot; z) \in S'(\mathbb{R}^3)$ for each $x \in \mathbb{R}^3_+$. We note that this depends on $x$ being an interior point of $\mathbb{R}^3_+$: if $x_3 = 0^+$, then $\chi_+ (\cdot) \lambda_x (\cdot) H(x, \cdot; z)$. 

27
does not exist in general: e.g., in (2.10) the singular integrals won’t exist, in general, if the integration in $I \ast \psi(x)$ goes only over the hemisphere $S^2_+$. Thus, $\chi_+(\cdot)H(x, \cdot; z)$ is a family of elements of $\mathcal{S}'(\mathbb{R}^3)$ depending continuously on $x \in \mathbb{R}^3_+$; but the limit as $x_3 \to 0$ of the family does not exist in general. (This illustrates the fact that no meaning can be assigned to the product of two distributions whose singular supports intersect.)

For $y \in \mathbb{R}^3_-$, $x \in \mathbb{R}^3_+$ $\chi_-(\cdot)I(x, \cdot; z)$ is a smooth function and so in $\mathcal{S}'(\mathbb{R}^3)$. Thus, $\chi_+(\cdot)I(x, \cdot; z) - \chi_-(\cdot)I(x, \cdot; z)$ is also in $\mathcal{S}'(\mathbb{R}^3)$. We wish to compute $\Phi_y^* \chi_+ I(x, \cdot; z)$.

As above in (3.3), using the fact that $I(x, \cdot; z)$ is the limit in $\mathcal{S}'$ of $I_h(x, \cdot; z)$, we compute

$$\Phi_y^* I(x, \xi, y_3; z) = (2\pi)^{-1/2} \exp(i x' \xi) \int_{-\infty}^{\infty} \exp[i(x_3 - y_3)\rho] \hat{I}(\eta) d\rho.$$  

For $x_3 \in \mathbb{R}_+, y_3 \in \mathbb{R}_-(x_3 - y_3) > 0$, so this can be evaluated by the residue theorem in $\mathbb{C}_+$. Taking the one-dimensional Fourier transform on $y_3$, we obtain

$$\Phi_y^* \chi_- I(x, \eta; z)$$

$$= (2\pi)^{-3/2} \exp(i x' \xi)(c_\rho^2|\eta|^2 - z)^{-1}(2\tau_p)^{-1} \exp(i\tau_p x_3)$$

$$(\rho + \tau_p)P(\xi, \tau_p) + (c_\rho^2|\eta|^2 - z)^{-1}(2\tau_s)^{-1} \exp(i\tau_s x_3)$$

$$(\rho + \tau_s)S(\xi, \tau_s).$$

(3.12)
Using the fact that \( \tau_{p,s}(\xi, \bar{z}) = -\tau_{p,s}(\xi, z) \), from (3.5), (3.6), (3.8), (3.9) it is straightforward to verify that

\[
\exp[-i\xi(x' + y')]\Phi_2 R(\xi, y_3, x; \bar{z}) = \Phi_2 R(\xi, x_3, y; z).
\]

Hence,

\[
\Phi^\ast y R(x, \xi, y_3; z) = \Phi^\ast y R(\xi, y_3, x; \bar{z}) = \exp[i(x' + y')]\Phi_2 R(\xi, x_3, y; z),
\]

so that from (3.8) we compute

\[
-(2\pi)^{3/2}\exp(i x' \xi) \Phi^\ast y \chi_+ R(x, \eta; z) =
\]

\[
(c_2^2|\eta|^2 - z)^{-1}(2\tau_p)^{-1}(\rho - \tau_p) [R_{pp} \exp(i\tau_p x_3) + R_{sp} \exp(i\tau_s x_3)] +
\]

\[
(c_2^2|\eta|^2 - z)^{-1}(2\tau_s)^{-1}(\rho - \tau_s) [R_{ss} \exp(i\tau_s x_3) + R_{ps} \exp(i\tau_p x_3)],
\]

\[
R_{pp} = c_p^2 z^{-1} r_{pp} P_+^t P_-, \quad R_{sp} = c_p^2 z^{-1} r_{sp} V_+^t V_-, \quad R_{ss} = c_s^2 z^{-1} r_{ss} V_+^t V_+ V_-, \quad (3.13)
\]

where \( r_{pp}, V_+ \), etc. are given in (3.5), (3.9).

Thus, for \( x \in \mathbb{R}_+^3, \chi_+ = \chi_+(y_3) \) in \( \mathcal{S}'(\mathbb{R}^3) \) there exists

\[
\Phi^\ast y \chi_+ H(x, \eta; z) = \Phi^\ast y \chi_+ I(x, \eta; z) - \Phi^\ast y \chi_+ R(x, \eta; z)
\]

which by (2.11), (3.12), (3.13) is given by

\[
\Phi^\ast y \chi_+ H(x, \eta; z) = (c_p^2|\eta|^2 - z)^{-1}\Psi_p(x, \eta; z) +
\]

\[
(c_s^2|\eta|^2 - z)^{-1}\Psi_s(x, \eta; z),
\]
\[ \Psi_p(x, \eta; z) = (2\pi)^{-3/2} \exp(ix'\xi) \{ \exp(i\rho x_3) P(\omega) - (2\tau_p)^{-1}(\rho + \tau_p) \exp(i\tau_p x_3) P(\xi, \tau_p) + (2\tau_p)^{-1}(\rho - \tau_p) \exp(i\tau_p x_3) R_{pp} + \exp(i\tau_p x_3) R_{sp} \} \}, \quad (3.14) \]

\[ \Psi_s(x, \eta; z) = (2\pi)^{-3/2} \exp(ix'\xi) \{ \exp(i\rho x_3) S(\omega) - (2\tau_s)^{-1}(\rho + \tau_s) \exp(i\tau_s x_3) S(\xi, \tau_s) + (2\tau_s)^{-1}(\rho - \tau_s) \exp(i\tau_s x_3) R_{ss} + \exp(i\tau_p x_3) R_{ps} \} \}. \]

We note that

\[ [M(D) - zI] \Psi_{p,s}(x, \eta; z) = (c_{p,s}^2 |\eta|^2 - z)(2\pi)^{-3/2} e^{ix\eta} Q(\omega), Q = P, S. \quad (3.15) \]

## 4 Solution of the Time-Dependent Problem in Terms of Generalized Eigenfunctions

In this section we obtain a representation of the solution of problem (2.17) in terms of an expansion in generalized eigenfunctions. This also affords a representation of the resolvent of \( M \) which is used in §§6-8 to establish the principle of limiting absorption.
With $\Psi(x, \eta; z)$ of (3.14) for $f \in \mathcal{D}(\mathbb{R}^3_+)$ we define

$$\Psi(z)f(\eta) = \int \overline{\Psi(x, \eta; z)}f(x)dx = \int \overline{[\Phi_y^*\chi_+H(x, \eta; z)]}f(x)dx.$$  

(4.1)

**Lemma 4.1** $\Psi(z)f(\eta) = \Phi\chi_+H(z)f(\eta)$.

**Proof.** For $\psi \in L_2(\mathbb{R}^3)$ let $\psi_+ = \chi_+\psi$, and let $\{\psi_n\} \subset \mathcal{D}(\mathbb{R}^3)$ be such that $\chi_+\psi_n \in \mathcal{D}(\mathbb{R}^3_+)$, $\chi_+\psi_n \to \psi_+$ in $L_2(\mathbb{R}^3_+)$, and $\psi_n \to \psi$ in $L_2(\mathbb{R}^3)$. Then denoting by $\langle \cdot, \cdot \rangle$ the inner product in $L_2(\mathbb{R}^3)$,

$$\langle \Phi\psi, \Phi\chi_+H(z)f \rangle = \langle \psi, \chi_+H(z)f \rangle = (\psi_+, H(z)f) = (H(z)\psi_+, f) = \lim(H(\cdot, y; z)\chi_+(y)\psi_n(y), f) = \lim\langle \chi_+(y)H(\cdot, y; z)\psi_n, f \rangle = \lim\langle \Phi_y^*\chi_+H(\cdot, \eta; z)\Phi\psi_n, f \rangle = \lim\langle \Phi\psi_n, \overline{[\Phi_y^*\chi_+H(\cdot, \eta; z)]}f \rangle = \langle \Phi\psi, \overline{[\Phi_y^*\chi_+H(\cdot, \eta; z)]}f \rangle.$$

Now let $f, g \in \mathcal{D}(\mathbb{R}^3_+)$, and let $e(\cdot)$ be the spectral measure of $M$ in $\mathcal{H}$. Setting $e(N) = e(0, N)$, by Stone’s formula [21], the resolvent identity, (4.1), Lemma 4.1, and the fact that $\lambda \to \Psi(\lambda \pm i\kappa)h, h = f, g$, is a continuous, $L_2$-valued function of $\lambda, \kappa > 0$, we have
\[ (f, e(N)g) = \]
\[ = \lim_{\kappa \to 0} (2\pi i)^{-1} \int_0^N (f, [H(\lambda + i\kappa) - H(\lambda - i\kappa)]g) d\lambda \]
\[ = \lim_{\kappa \to 0} \pi^{-1} \int_0^N \kappa (f, H(\lambda \pm i\kappa) H(\lambda \mp i\kappa) g) d\lambda \]
\[ = \lim_{\kappa \to 0} \pi^{-1} \int_0^N \kappa (H(\lambda \mp i\kappa) f, H(\lambda \mp i\kappa) g) d\lambda \]
\[ = \lim_{\kappa \to 0} \pi^{-1} \int_0^N \kappa \langle \Phi \chi + H(\lambda \mp i\kappa) f, \Phi \chi + H(\lambda \mp i\kappa) g \rangle d\lambda \]
\[ = \lim_{\kappa \to 0} \pi^{-1} \int_0^N \kappa \langle \Psi(\lambda \pm i\kappa) f, \Psi(\lambda \pm i\kappa) g \rangle d\lambda \]
\[ = \lim_{\kappa \to 0} \pi^{-1} \int_0^N \kappa d\lambda \int_{\mathbb{R}^3} i \langle \Psi(\lambda \pm i\kappa) f(\eta) \rangle \Psi(\lambda \pm i\kappa) g(\eta) d\eta \]
\[ = \lim_{\kappa \to 0} \pi^{-1} \int_{\mathbb{R}^3} d\eta \int_0^N \kappa i \langle \Psi(\lambda \pm i\kappa) f(\eta) \rangle \Psi(\lambda \pm i\kappa) g(\eta) d\lambda. \]

We now pass to the limit under the last integral and evaluate the result. The justification and technique for doing this are essentially the same as in [5, 6, 14] and are not reproduced.
here. To state the result we must compute the generalized eigenfunctions of $M$ from (3.9), (3.13), and (3.14). Details of such a computation can be found in the aforecited references.

Denoting the characteristic functions of the hemispheres $S^2_\pm$ by $\chi_\pm(q_3)$, from (3.14) we have with the notation of (2.5),

$$\eta = (\xi, \rho) = |\eta|q, \quad q \in S^2, \quad \bar{q} = (q', -q_3), \quad \bar{\eta} = (\xi, -\rho)$$

$$\Psi^\pm_s(x, \eta) \equiv \Psi_s(x, \eta; c^2_s|\eta|^2 \pm i0)$$

$$= \Psi^\pm_{sv}(x, \eta) + \Psi^\pm_{sh}(x, \eta),$$

$$\Psi^\pm_{sh}(x, \eta) = \chi_\pm(q_3)(2\pi)^{-3/2}[\exp(ix\eta) + \exp(ix\bar{\eta})]h(q) \otimes h(q)$$

$$\Psi^\pm_{sv}(x, \eta) = \chi_\pm(q_3)(2\pi)^{-3/2}[\exp(ix\eta)v(q) - \exp(ix\bar{\eta})r^\pm_{ss}(q)v(\bar{q})$$

$$- \exp(ix\eta^\pm_s r^\pm_{ps}(q)\theta^\pm(q))] \otimes v(q),$$

$$r^\pm_{ss}(q) = \tilde{\Delta}^\pm_s(q)/\Delta^\pm_s(q), \quad (4.3)$$

$$r^\pm_{ps}(q) = \pm 4n|q_3| |q'|(q_3^2 - |q'|^2)/\Delta^\pm_s(q),$$

$$\Delta^\pm_s(q) = (q_3^2 - |q'|^2) \pm 4n|q'|^2|q_3|\theta^\pm_3,$$

$$\tilde{\Delta}^\pm_s(q) = (q_3^2 - |q'|^2) \mp 4n|q'|^2|q_3|\theta^\pm_3,$$

$$\theta^\pm(q) = t(\theta_1, \theta_2, \theta^\pm_3) \equiv t(n^{-1}q_1, n^{-1}q_2, \theta^\pm_3),$$

$$\theta^\pm_3(q) = \pm \chi_{0,n}(|q'|)\sqrt{(1 - n^{-2}|q'|^2)} +$$

$$i\chi_{n,1}(|q'|)\sqrt{(n^{-2}|q'|^2 - 1)},$$

$$33$$
\[
\eta^\pm_s = |\eta|(q', n\theta^\pm_s), \quad n = c_s/c_p.
\]

From (3.15) and by direct verification
\[
M(D)\Psi^\pm_{sv}(x, \eta) = c_s^2|\eta|^2\Psi^\pm_{sv}(x, \eta),
\]
\[
M(D)\Psi^\pm_{sh}(x, \eta) = c_s^2|\eta|^2\Psi^\pm_{sh}(x, \eta),
\]
\[
B(D)\Psi^\pm_{sv}(x', 0, \eta) = B(D)\Psi^\pm_{sh}(x', 0, \eta) = 0,
\]
so \(\Psi^\pm_{sv}(x, \eta)\) are generalized eigenfunctions of \(M\) corresponding to shear modes, and \(\Psi^\pm_{sv}(x, \eta)\) and \(\Psi^\pm_{sh}(x, \eta)\) are, respectively, the vertical and horizontal components. The second component corresponds to the reflected \(S\) mode, while the third component of \(\Psi^\pm_{sv}(x, \eta)\),
\[
\exp(ix\eta^\pm_s) r^\pm_{ps}(q) \theta^\pm(q),
\]
is the reflected pressure mode due to the incident shear mode.

We note that the phase \(\eta^\pm_s\) has a purely imaginary third component for \(|q'| > n = c_s/c_p < 1/\sqrt{2}\).

Further, from (3.14), retaining the notation of (4.3),
\[
\Psi^\pm_p(x, \eta) \equiv \Psi_p(x, \eta; c_p^2|\eta|^2 \pm i0)
\]
\[
= \chi_\pm(q_3)(2\pi)^{-3/2}\exp(\pm i\eta)q - \exp(\pm i\eta) r^\pm_{pp}(q)q -
\]
\[
- \exp(\pm i\eta) v^\pm(\phi) q, \quad r^\pm_{pp}(q) = \tilde{\Delta}(q)/\Delta(q),
\]

34
\[ r_{sp}^\pm(q) = \mp r_{sp}(q), \quad r_{sp}(q) = 4|q_3| |q'| (\phi_3^2 - |\phi'|^2) \]

\[ \phi' = nq', \quad \phi_3 = \sqrt{(1 - |\phi'|^2)}, \]

\[ \eta_p^\pm = n^{-1}|\eta|\phi^\pm \equiv n^{-1}|\eta|(\phi', \pm \phi_3) = |\eta|(q', \pm n^{-1}\phi_3 \phi) \]

\[ \Delta_p(q) = (\phi_3^2 - |\phi'|^2) + 4n|\phi'|^2 \phi_3 |q_3|, \]

\[ \tilde{\Delta}_p(q) = (\phi_3^2 - |\phi'|^2) - 4n|\phi'|^2 \phi_3 |q_3|, \]

\[ v^\pm(\phi) = t(\pm \phi_3 \phi_1, \pm \phi_3 \phi_1, -|\phi'|^2). \]

As above,

\[ M(D)\Psi_p^\pm(x, \eta) = c_p^2 |\eta|^2 \Psi_p^\pm(x, \eta), \]

\[ B(D)\Psi_p^\pm(x', 0, \eta) = 0, \]

so \( \Psi_p^\pm(x, \eta) \) are generalized eigenfunctions corresponding to a \( P \) mode. Here \( q \in S^2 \) is the direction of the incident \( P \) mode, \( \bar{q} \in S^2 \) is that of the reflected \( P \) mode, and \( \phi \) is that of the reflected \( SP \) mode – the \( S \) mode due to the incident \( P \) mode.

There is a further contribution to the limit (4.2) due to the pole of the reflection and transmission coefficients (3.9) entering in (3.14), i.e., due to the real zero of \( \Delta(\xi; z) \). This zero gives rise to the Rayleigh or surface mode.

For each \( \mathbb{R}^2 \ni \xi \neq 0 \) \( \Delta(\xi; z) \) has a simple zero [4, 14]

\[ R^2(\xi) = R_0^2 c_\eta^2 |\xi|^2, \quad R_0 \in (0, 1), \]
\[(2 - R_0^2)^2 = 4(1 - R_0^2)^{1/2}(1 - n^2 R_0^2)^{1/2}\]

(4.6)

with

\[\partial_z \Delta(\xi; z)|_{z=R} = 2c_s^{-2}|\xi|^2\]

(4.7)

\[\alpha(R_0)\alpha(R_0) = (1-R_0^2)^{-1/2}(1-n^2 R_0^2)[1+n^2-2n^2 R_0^2-(2-R_0^2)/4] > 0.\]

(Note that \(R^2(\xi)\) lies to the left of the least branch point \(c_s^2|\xi|^2\).) Passing to the limit \(z = \lambda + i\kappa \rightarrow R^2\) in (4.2) and then integrating on \(\rho\) [4, 5, 14], we see that the Rayleigh mode has the form

\[\Sigma(x, \xi) = (2\pi)^{-1}\lambda(R_0)^{1/2}|\xi|^2 \exp(i x' \xi) \gamma(\xi, x_3),\]

\[\gamma(\xi, x_3) = (1-n^2 R_0^2)^{-1/2}(R_0^2 - 2)\pi(\beta) \exp[-x_3|\xi|\sqrt{(1-n^2 R_0^2)}] \]

\[\quad - 2i\sigma(\beta) \exp[-x_3|\xi|\sqrt{(1-R_0^2)}],\]

\[\pi(\beta) = t(\beta_1, \beta_2, i\sqrt{(1-n^2 R_0^2)}),\]

(4.8)

\[\sigma(\beta) = t(i\beta_1\sqrt{(1-R_0^2)}, i\beta_2\sqrt{(1-R_0^2)}, -1),\]

\[\beta_i = \xi_i/|\xi|, \quad i = 1, 2,\]

\[\lambda(R_0) = 2^{-1}\tau(R_0)\alpha(R_0)^{-2} R_0^{-4},\]

\[\tau(R_0) = (1-R_0^2)^{-1/2}(1-R_0^2)(2-n^2 R_0^2) +\]

36
Using (4.6), (4.7), it can be verified directly (but perhaps not so easily [4]) that

\[ \tau(R_0)^2 = \alpha(R_0) R_0^4 (1 - n^2 R_0^2). \]  

(4.9)

This may then be used in \( \lambda(R_0) \) in place of the lengthy expression above.

It follows by direct verification that

\[ M(D) \Sigma(x, \xi) = R^2(\xi) \Sigma(x, \xi) = c_s^2 R_0^2 |\xi|^2 \Sigma(x, \xi) \]  

(4.10)

\[ B(D) \Sigma(x', 0, \xi) = 0. \]

Thus, \( \Sigma(x, \xi) \) is a generalized eigenfunction of \( M \) corresponding to surface modes. Multiplied by \( \exp(-i\nu t) \), they propagate as cylindrical modes along the boundary \( \{x_3 = 0\} \) while decaying exponentially in the normal direction.

**Lemma 4.2** Let \( f \in D(\mathbb{R}^3_+) \). Then

\[ \Psi^\pm_{p,s} f(\eta) = \int i \Psi^\pm_{p,s}(x, \eta) f(x) dx, \]

(4.11)

\[ \Sigma f(\xi) = \int i \Sigma(x, \xi) f(x) dx \]

extend to bounded mappings from \( \mathcal{H} \) to \( L_2(\mathbb{R}^3, \mathbb{C}^3) \) and \( L_2(\mathbb{R}^2, \mathbb{C}^3) \) respectively. Their adjoints are

\[ \Psi^*_{p,s} g(x) = \int \Psi^\pm_{p,s}(x, \eta) g(\eta) d\eta, \]
\[ \Sigma^* h(x) = \int \Sigma(x, \xi) h(\xi) d\xi. \]

The proof proceeds by direct verification using (4.3), (4.5), (4.8) (cf., e.g., [3, 4]).

**Remark 4.1** In the proof of Theorem 4.1 below we shall need to know the following facts which follow directly from (4.3), (4.5), (4.8), (4.11) and (4.12):

\[ \Sigma^* h(x) = \lambda(R_0) \Phi_2^2[|\cdot|^{1/2} \gamma(\cdot, x_3) h(\cdot)(x')], \]

\[ \Psi^\pm f(\eta) = \chi^\pm(q_3) q \otimes [q \Phi f(\eta) - r_{pp}(q) \bar{q} \Phi f(\bar{\eta}) - r_{ss}(q) v^\pm(\phi) \Phi f(\eta^\pm)], \]

\[ \Psi^\pm f(\eta) = \chi^\pm(q_3) \{h(q) \otimes h(q)[\Phi f(\eta) + \Phi f(\bar{\eta})] + v(q) \otimes [v(q) \Phi f(\eta) - r_{ss}(q) v^\pm(q) \Phi f(\eta^\pm)] \}
- r_{pp}(q) \theta^{\pm}(q) \Phi f(\eta^\pm))], \]

\[ \Phi f(\eta^\pm_p) = (2\pi)^{-1/2} \int \exp(\mp in^{-1} |\eta| \phi_3 x_3) \Phi_2 f(\xi, x_3) dx_3, \]

\[ \Phi f(\eta^\pm_s) = (2\pi)^{-1/2} \int \exp(in |\eta| \tilde{\theta}_3 x_3) \Phi_2 f(\xi, x_3) dx_3. \]

Denoting now by \( \chi_{p,N'} \chi_{s,N'} \) and \( \chi_{R,N} \) the characteristic functions of the respective sets \( \{\eta : c_p^2 |\eta|^2 \in (0, N)\} \), \( \{\eta : c_s^2 |\eta|^2 \in (0, N)\} \), and \( \{\xi : R^2(\xi) = c_s^2 |\xi|^2 R_0^2 \in (0, N)\} \), in
terms of (4.11), (4.12) the expression (4.2) can now be written ($\Psi_{p,s}$ denotes either $\Psi^+_{p,s}$ or $\Psi^-_{p,s}$)
\[(f, e(N)g) = (\Psi_p f, \chi_{p,N} \Psi_p g) + (\Psi_s f, \chi_{s,N} \Psi_s g) + (\Sigma f, \chi_{R,N} \Sigma g). \]
(4.13)

**Theorem 4.1** The spectral measure $e(N) = e(0, N)$ of $M$ is given by
\[
e(N) = \Psi_p^* \chi_{p,N} \Psi_p + \Psi_s^* \chi_{s,N} \Psi_s + \Sigma^* \chi_{R,N} \Sigma \]
(4.14)
which for $N \to \infty$ gives the Parseval identity for $M$:
\[
I = \Psi_p^* \Psi_p + \Psi_s^* \Psi_s + \Sigma^* \Sigma.
\]
(4.15)

For $f \in \mathcal{D}(M)$
\[
\Psi_{p,s} M f(\eta) = c^2_{p,s} |\eta|^2 \Psi_{p,s} f(\eta),
\]
(4.16)
\[
\Sigma M f(\xi) = R(\xi)^2 \Sigma f(\xi),
\]
and
\[
M f = \Psi_p^* c_p^2 |\cdot|^2 \Psi_p f + \Psi_s^* c_s^2 |\cdot|^2 \Psi_s f + \Sigma^* R(\cdot)^2 \Sigma f.
\]
(4.17)
The operators $\Psi_p^* \Psi_p + \Psi_s^* \Psi_s$ and $\Sigma^* \Sigma$ are mutually orthogonal orthoprojectors in $\mathcal{H}$.

**Proof.** Equality (4.14) is just (4.13). To prove (4.16), let $f \in \mathcal{D}(M)$ and recall that $B(D) = \mathcal{A}(n) E^{-1} \mathcal{A}(D), n =$
(0, 0, 1) (see (3.1). All three relations (4.16) are verified in the same way simply by integrating by parts and recalling that \( f \) and the generalized eigenfunctions satisfy the boundary conditions. Consider, for example,

\[
\Psi_p M(D)f(\eta) = \Psi_p^t A(D) E^{-1} A(D)f(\eta)
\]

\[
= \int_{\mathbb{R}^3_+} i^t \Psi(x, \eta) \cdot A(D) E^{-1} A(D) f(x) dx =
\]

\[
\int_{\mathbb{R}^3_+} i^t [A(D) \Psi_p(x, \eta)] E^{-1} A(D) f(x) dx +
\]

\[
+ \int_{\mathbb{R}^2} i^t \Psi_p(x', 0, \eta) B(D) f(x', 0) dx'
\]

\[
= \int_{\mathbb{R}^3_+} i^t [M(D) \Psi_p(x, \eta)] f(x) dx +
\]

\[
\int_{\mathbb{R}^2} i^t [B(D) \Psi_p(x', 0, \eta)] f(x', 0) dx' = c^2_p |\eta|^2 \Psi_p f(\eta).
\]

The result for any \( f \in \mathcal{D}(M) \) now follows in an obvious way (see, e.g., [5], Lemma 4.13). Equality (4.17) follows formally from (4.15) and rigorously from (4.14) and the ensuing functional calculus [14]. From (4.8), (4.10), (4.12)

\[
M(D) \Sigma^* h(\xi) = \lambda(R_0)^{1/2} \Phi_2^* || \cdot ||^{1/2} \gamma(\cdot, x_3) R^2(\cdot) h(\cdot))(x').
\]

40
From Remark 4.1

\[ \Psi_{p,s} M(D) \Sigma^* h(\eta) = R(\xi)^2 \Psi_{p,s} \Sigma^* h(\eta). \]

From (4.16)

\[ \Psi_{p,s} M(D) \Sigma^* h(\eta) = c_{p,s}^2 |\eta|^2 \Psi_{p,s} \Sigma^* h(\eta). \]

Subtraction gives

\[ [c_{p,s}^2 |\eta|^2 - R(\xi)^2] \Psi_{p,s} \Sigma^* \Sigma h(\eta) = 0 \]

which implies the orthogonality relation. The fact that the operators are projections follows from this and (4.15) (it can be verified by direct computation that \( \Sigma^* \Sigma \) is a projection [4]).

**Remark 4.2** It was stated in [14] that the operators \( \Psi^* \Psi \) and \( \Psi^*_s \Psi_s \) are mutually orthogonal by appeal to a proof of a similar result in [13]. That proof was subsequently found to be incorrect, and a valid proof for the case of Maxwell’s equations was given in [5]. However, that proof does not work in the present situation. We very much doubt if such an assertion is true (see the form of the asymptotics of the steady-state waves in §§7-8). Be this as it may, it has no bearing on the present work.

We now recall from §2 the energy space \( \mathcal{H}_M \) associated with \( M \).
Theorem 4.2 In terms of the mappings
\[ \tilde{\Psi}_{\pm} : \mathcal{H}_M \rightarrow L_2(\mathbb{R}^3; \mathbb{C}^3). \] (4.18)
\[ \tilde{\Sigma}_{\pm} : \mathcal{H}_M \rightarrow L_2(\mathbb{R}^2; \mathbb{C}^3) \]
defined by
\[ \tilde{\Psi}_{\pm,p,s}f = 2^{-1/2}[c_{p,s} \cdot |I_3, \mp iI_3] \begin{bmatrix} \Psi_{p,s}f_1 \\ \Psi_{p,s}f_2 \end{bmatrix}, \] (4.19)
\[ \tilde{\Sigma}_{\pm}f = 2^{-1/2}[R(\cdot)I_3, \mp iI_3] \begin{bmatrix} \Sigma f_1 \\ \Sigma f_2 \end{bmatrix} \]
with adjoints
\[ \tilde{\Psi}_{\pm,p,s}^*g = 2^{-1/2} \begin{bmatrix} \Psi_{p,s}^*(c_{p,s} \cdot |^{-1})g \\ \pm i\Psi_{p,s}^*g \end{bmatrix}, \quad g \in L_2(\mathbb{R}^3, \mathbb{C}^3), \] (4.20)
\[ \tilde{\Sigma}_{\pm}^*h = 2^{-1/2} \begin{bmatrix} \Sigma^* R(\cdot)^{-1}h \\ \pm i\Sigma^*h \end{bmatrix}, \quad h \in L_2(\mathbb{R}^2, \mathbb{C}^3), \]
the solution of the initial boundary value problem
\[ \partial_t^2 f_1(x, t) + M f_1(x, t) = 0, \quad x \in \mathbb{R}^3_+, \quad t \in \mathbb{R}, \]
\[ f_1(x, 0) = f^0_1(x), \quad \partial_t f_1(x, 0) = f^0_2(\cdot), \] (4.21)
\[ B f_1(x', 0, t) = 0 \]
is given by

\[ f_1(x, t) = [U(t)f^0]_1(x), \quad f^0 = \begin{bmatrix} f_1^0 \\ f_2^0 \end{bmatrix}, \]

where \( U(t) = \exp(i\mathcal{M}t) \) is the unitary group in the energy space \( \mathcal{H}_M \) generated by the selfadjoint operator \( \mathcal{M} \) of Theorem 2.1 and has the representation

\[ U(t)f^0 = \sum_{j=\pm 1} \{ \tilde{\Psi}^*_j \exp(ijc_\parallel \cdot |t|)\tilde{\Psi}_j f^0 + \tilde{\Psi}_j \exp(ijc_\perp \cdot |t|)\tilde{\Psi}^*_j f^0 \}
+ \tilde{\Sigma}^*_j \exp(ijR(\cdot)t)\tilde{\Sigma}_j f^0 \} \quad (4.22) \]

The mappings \( \tilde{\Psi}_{\pm p,s}, \tilde{\Sigma}_{\pm} \) are obtained on the basis of (2.13), (4.14); here \( \tilde{\Psi}_{\pm p,s} \) denotes four mappings each of which contains one of the two mappings \( \Psi_{\pm}^\parallel, \Psi_{\pm}^\perp \). The representation of \( U(t) \) follows from (2.13), (2.14), and (4.14).

5 The Steady-State Problem and the Resolvent in Terms of Generalized Eigenfunctions

The Lamb problem for time-harmonic solutions is that of finding solutions of the problem

\[
\begin{align*}
\partial_t^2 u(x, t) + M(D)u(x, t) &= f(x)\exp(-i\nu t) \\
B(D)u(x', 0, t) &= 0, \quad \nu \in \mathbb{R}_+, \quad f \in \mathcal{D}(\mathbb{R}^3_+), \quad x \in (\mathbb{R}_+^3)
\end{align*}
\]
If \( v(x) \) is a solution of the steady-state problem

\[
[M(D) - \nu^2 I]v(x) = f(x),
\]

\[
B(D)v(x',0) = 0, \quad \nu > 0, \quad (5.2)
\]

then \( u(x,t) = \exp(i\nu t)v(x) \) will be a time-harmonic solution.

To proceed we set \( z = (\nu \pm i\epsilon)^2, \epsilon \in (0,\epsilon_0] \) and write the resolvent for \( M \) in terms of generalized eigenfunctions in the form

\[
v(x; z) \equiv H(z)f(x) = \Psi^*(c_p^2 \cdot |^2 - z)^{-1}\Psi_p f(x) + \Psi^*(c_s^2 \cdot |^2 - z)^{-1}\Psi_s f(x) + \Sigma^*(R^2(\cdot) - z)^{-1}\Sigma f(x) \quad (5.3)
\]

\[
\equiv v_p(x; z) + v_s(x; z) + v_R(x; z).
\]

We must now demonstrate that each of these terms has a limit as \( \epsilon \to 0 \) – the principle of limit absorption. Because of (5.2) with \( \nu^2 \) replaced by \( z = (\nu \pm i\epsilon)^2 \), the resulting limit will automatically be a solution of (5.2).

6 The Steady-State Rayleigh Wave

In this section we derive the surface-wave component of the steady-state solution and find its asymptotics as \( |x| \to \infty \).

To simplify notation we consider only \( D^\beta v_R \) for \( |\beta| = 0 \), but
it is obvious that an expression of exactly the same form holds for $|\beta| > 0$.

With the notation of (4.8), (5.3)

$$v_R(x; z) = \Sigma^*(R^2(\cdot) - z)^{-1}\Sigma f(x)$$

$$= \lambda(R_0)(2\pi)^{-1} \int_{\mathbb{R}^2} d\xi [R(\xi)^2 - z]^{-1} \exp(ix'\xi)\xi|\gamma(\xi, x_3) \times$$

$$\int_0^\infty \int \bar{t} \gamma(\xi, y_3)\Phi_2 f(\xi, y_3) dy_3, \quad (6.1)$$

where $z = (\nu \pm i\epsilon)^2, \nu \in (\nu_0 - \delta, \nu_0 + \delta), \nu_0 - 4\delta > 0, \epsilon \in (0, \epsilon_0]$. Let $\psi \in C^\infty_0(\mathbb{R})$, supp $\psi \subset \{|c_s R_0 r - \nu_0| < 4\delta\} \subset (0, \infty)$, $\psi(r) = 1$ for $r \in \{|c_s R_0 r - \nu_0| < 3\delta\}, \psi(r) = 0$ for $r \in \{|c_s R_0 r - \nu_0| > 4\delta\}$. We define $\chi \in C^\infty_0(\mathbb{R}^2 \setminus \{0\})$ by $\chi(\xi) = \chi(rq) = \psi(r), q \in S^1$. Then

$$v_R(x; z) = I(x; z) + J(x; z), \quad (6.2)$$

where, as in [16], formulas (3.5)-(3.8),

$$J(x; z) =$$

$$= \lambda(R_0)(2\pi)^{-1} \int d\xi [R^2 - z]^{-1} \exp(ix'\xi)[1 - \chi(\xi)]\xi|\gamma(\xi, x_3) \times$$

$$\int \bar{t} \gamma(\xi, y_3)\Phi_2 f(\xi, y_3) dy_3 \quad (6.3)$$

$$= 0(|x|^{-2}),$$

45
\[
I(x; z) = (2\pi)^{-2} \lambda(R_0) \int_{\mathbb{R}^2} d\xi \int_{\mathbb{R}^3_+} \exp(ix'\xi)(R^2 - z)^{-1}\chi(\xi)|\gamma(\xi, x_3) \times \]
\[
\gamma(\xi, y_3) \exp(-iy'\xi)f(y)dy
\]
\[
= (2\pi)^{-2} \lambda(R_0) \int_{\mathbb{R}^3} D(x, y; z)f(y)dy, \quad (6.4)
\]
\[
D(x, y; z) = \int_0^\infty (c^2 R_0^2 r^2 - z)^{-1}\psi(r)r^2dr \int_{S^1} \gamma(\xi, x_3)^f\gamma(\xi, y_3) \times \exp[i(x' - y')\xi]d\phi
\]
\[
= (r \cos \phi, r \sin \phi) \equiv rq, q \in S^1.
\]
Setting \(|\xi| = r\), from (4.8) we now compute
\[
\gamma(\xi, x_3)^f\gamma(x, y_3) = (R_0^2 - 2)^2(1 - n^2 R_0^2)^{-1}\pi(q)^f\pi(q)
\]
\[
\exp[-(x_3 + y_3)r \sqrt{(1 - n^2 R_0^2)}] + (6.5)
\]
\[
2i(R_0^2 - 2)(1 - n^2 R_0^2)^{-1/2}\pi(q)^f\sigma(q)
\]
\[
\exp[-x_3\sqrt{(1 - n^2 R_0^2)} - y_3r \sqrt{(1 - R_0^2)}] -
\]
\[-2i(R_0^2 - 2)(1 - n^2 R_0^2)^{-1/2}\sigma(q)^f\pi(q)
\]
\[
\exp[-x_3r \sqrt{(1 - R_0^2)} - y_3r \sqrt{(1 - n^2 R_0^2)}] +
\]
46
We define
\[
\phi(r) = r^{3/2} \psi(r)/(c_s R_0 r + \nu \pm i0) \tag{6.6}
\]
and set
\[
\begin{align*}
\phi^1(r) &= \phi(r) \exp[-(x_3 + y_3)r \sqrt{1 - n^2 R_0^2}] \\
\phi^2(r) &= \phi(r) \exp[-x_3 r \sqrt{1 - n^2 R_0^2} - y_3 r \sqrt{1 - R_0^2}] \\
\phi^3(r) &= \phi(r) \exp[-x_3 r \sqrt{1 - R_0^2} - y_3 r \sqrt{1 - n^2 R_0^2}] \tag{6.7} \\
\phi^4(r) &= \phi(r) \exp[-(x_3 + y_3)r \sqrt{1 - R_0^2}].
\end{align*}
\]
Then $D(x, y; z)$ of (6.4) can be written
\[
D(x, y; z) = \\
= \int_0^\infty \left[ c_s R_0 - (\nu \pm i\epsilon) \right]^{-1} dr \left\{ \left[ (R_0^2 - 2)/(1 - n^2 R_0^2) \right] \right\}
\]
\[
\phi^1(r) \int_{S^1} \exp[i(x - y')rq] \pi(q)^\pi(q) d\phi + \tag{6.8}
\]
\[
[2i(2 - R_0^2)/\sqrt{1 - n^2 R_0^2}] \phi^2(r) \int_{S^1} \exp[i(x - y')rq] \pi^\pi d\phi +
\]
\[
[2i(2 - R_0^2)/\sqrt{1 - n^2 R_0^2}] \phi^3(r) \int_{S^1} \exp[i(x - y')rq] \sigma^\pi d\phi +
\]

47
\[ 4\phi^4(r) \int_{S^1} \exp[i(x' - y')rq]\sigma(q)\tilde{\sigma}(q)d\phi. \]

We now need the principle of stationary phase [16, 19]: for large \(|x|\), uniformly with respect to \(r\) in bounded intervals of \(\mathbb{R}_+\) and \(y'\) in compact sets, for a smooth scalar function \(\psi\) and a smooth matrix-valued function \(R\) with \(x' = |x'|\alpha, \alpha = (\cos \phi_x, \sin \phi_x)\)

\[ \int_{S^1} \exp(irx'y)\psi(r, y', q)R(q)d\phi = (2\pi)^{1/2}|rx'|^{-1/2} \times \]

\[ \sum_{j=\pm 1} \{ \exp(ijr|x'| - ij\pi/4)\psi(r, y, j\alpha)R(j\alpha) \} + q(rx'), \]

\[ |D^\beta q(x')| = 0(|x'|^{-3/2}), |\beta| \geq 0. \quad (6.9) \]

From (6.7), (6.8), (6.9), and a remainder estimate as in [11, 16] we have

\[ D(x, y; z) = \]

\[ = (2\pi)^{1/2}|x'|^{-1/2} \sum_{j=\pm 1} \exp(ij\pi/4) \{ [R_{0}^2 - 2]^2/(1 - n^2 R_{0}^2) \}
\]

\[ \pi(j\alpha)^{\bar{\tau}}\pi(j\alpha)I_{1}^{\pm}(t_j, \nu, \epsilon) + [2i(R_{0}^2 - 2)/\sqrt{(1 - n^2 R_{0}^2)}] \]

\[ \pi(j\alpha)^{\bar{\tau}}\sigma(j\alpha)I_{2}^{\pm}(t_j, \nu, \epsilon) - [2i(R_{0}^2 - 2)/\sqrt{(1 - n^2 R_{0}^2)}] \] \quad (6.10)

\[ \sigma(j\alpha)^{\bar{\tau}}\pi(j\alpha)I_{3}^{\pm}(t_j, \nu, \epsilon) + 4\sigma(j\alpha)^{\bar{\tau}}\sigma(j\alpha)I_{4}^{\pm}(t_j, \nu, \epsilon) \} + R(x), \]
\[ I^\pm_k(t_j, \nu, \epsilon) = \int_0^\infty \beta^k(r) \exp(irt_j/c_sR_0)[r - (\nu \pm i\epsilon)]^{-1}dr, \]

\[ \beta^k(r) = \phi^k(r/c_sR_0)(c_sR_0)^{-1}, k = 1, 2, 3, 4, \]

\[ |R(x)| \leq \text{const} \ (1 + x_3) \exp(-ax_3)|x'|^{-1-\kappa}, \ (0, 1/2) \ni \kappa \]

arbitrarily close to 1/2, \( a = a(\nu_0, \delta) > 0, \)

\[ t_j = j(|x'| - \alpha y'). \]

Using the Fourier transform in the usual way [16], we write (\( \chi^\pm \) are the characteristic functions of \( \mathbb{R}_\pm \))

\[ I^\pm_k(t_j, \nu, \epsilon) = \pm i(2\pi)^{1/2} \int_{-\infty}^\infty \chi^\pm(t_j/c_sR_0 - \tau) \exp[i(\nu \pm i\epsilon) \times (t_j/c_sR_0 - \tau)] \Phi^*_1 \beta^k(\tau) d\tau \] (6.11)

from which it follows that \( I^\pm_k(t_j, \nu, \epsilon) \) are bounded and continuous for \( x_3, y_3 \in \mathbb{R}^3_+, \ \nu \in [\nu_0 - \delta, \nu_0 + \delta], \ \epsilon \in [0, \epsilon_0] \) which with (6.3), (6.4), (6.10) establishes the principle of limiting absorption for \( v_R \). Computing in the usual manner [16], we obtain for \( |x'| \) large

\[ v_R(x; (\nu \pm 0)^2) = c^\pm(R_0, \nu)|x'|^{-1/2}\gamma^\pm(x) \int i\gamma^\pm(y)f(y)dy + R(x) \]
\[\gamma_\pm(x) = \exp(\pm i\nu |x'|/c_s R_0) \{(R_0^2 - 2) \sqrt{(1 - n^2 R_0^2)} \pi(\pm \alpha) \}
\]
\[\exp[-x_3 \nu \sqrt{(1 - n^2 R_0^2)/c_s R_0}] - 2i\sigma(\pm \alpha) \exp[-x_3 \nu \sqrt{(1 - R_0^2)/c_s R_0}],\]
\[c_\pm(R_0, \nu) = \pm i(2\pi)^{-1/2} \exp(\pm i\pi/4) \nu^{1/2}(c_s R_0)^{-5/2} \lambda(R_0),\]
\[\kappa' = |x'| \alpha, \quad \alpha = (\cos \phi_x, \sin \phi_x),\]
\[\pi(\alpha) = t(\alpha_1, \alpha_2, i \sqrt{(1 - n^2 R_0^2)}),\]
\[\sigma(\alpha) = t(i\alpha_1 \sqrt{(1 - R_0^2), i\alpha_2 \sqrt{(1 - R_0^2), -1}}),\]
\[|R(x)| \leq \text{const} (1 + x_3) \exp(-a x_3 |x'|^{1-\kappa'}, \kappa' \in (0, 1/2) \]
arbitrarily close to 1/2, \(a = a(\nu) > 0.\)

We check that
\[\begin{align*}
[M(D) - \nu^2 I]\gamma_\pm(x) &= 0, \\
B(D)\gamma_\pm(x', 0) &= 0,
\end{align*}\]
so that, in particular, \([M(D) - \nu^2 I]v_R \in L_2.\)

We thus see that in order \(|x'|^{-1/2} v_R(x; \nu + i0) \exp(-i\nu t)\)
is an outgoing cylindrical wave, while \(v_R(x; \nu - i0) \exp(-i\nu t)\)
is an incoming cylindrical wave. They both decay exponentially away from the boundary \(\{x_3 = 0\}\). The leading term
satisfies the homogeneous equation and the boundary condition to $0(|x'|^{-3/2})$.

Finally, in formulating uniqueness classes in §8, it is essential to observe the following fact: for all $\alpha \in S^1$ independent of $x'$

$$\pm \int_0^\infty \ell_\gamma(x) A(\alpha, 0) E_0^{-1} A(D) \gamma_\pm(x) dx_3 = \text{const} > 0. \quad (6.14)$$

The verification of this is simply a computation, but it requires some organization in order to make it amenable. We therefore present the proof in the appendix.

7 The Steady-State SH and PSV Waves

The component $v_S(x; (\nu \pm i0)^2)$ of the steady-state solution consists of two parts. The first is an SH wave due to the incident and reflected SH modes. The second is formed by an SV wave and a P wave due to incident and reflected in SV modes and reflected P modes; this part we call the PSV wave. In this section we derive these components of the solution and determine their asymptotic behavior as $|x| \to \infty$. To simplify notation the arguments are presented only for $D^\beta v_S$ with $|\beta| = 0$. It is obvious that entirely similar expressions hold for $|\beta| > 0$.

Below we shall need the principle of stationary phase in
the following two forms. Suppose $R(q)$ is a smooth, matrix-valued function on the unit sphere $S^2 \ni q, x = |x| \omega \in \mathbb{R}_+^3, y \in \mathbb{R}_+^3, r > 0$, and $\psi(r, y, q)$ is a smooth function. Then for large $|x|$, uniformly with respect to $r$ in bounded intervals of $\mathbb{R}_+$ and $y$ in compact sets,

$$
\int_{S^2} \exp(ir|x|)\psi(r, y, q)R(q)dS_q = 2\pi \sum_{j=\pm 1} \{\exp(ijr|x| - ij\pi/2) \}
$$

$$
\psi(r, y, j\omega)R(j\omega)|rx|^{-1}
$$

$$
+ q(rx),
$$

$$
D^\alpha q(x) = 0(|x|^{-2}) \text{ and smooth (see [18])}.
$$

Suppose for each $\omega \in S^2$ $F(\omega, q)$ has an isolated, non-degenerate critical point $q_c(\omega) \in S^2$, $[q_c(\omega)]_3 \neq 0$, $q_c(\omega)$ smooth. Let $\phi(q) \in C_0^\infty(S^2), q_c \in \text{supp } \phi, \phi(q_c) = 1$, supp $\phi \cap \{q_3 = 0\} = \emptyset$, and let $R(q)$ be a smooth matrix-valued function. Then

$$
I(r, x, y) = \int_{S^2} \phi(q) \exp[i\pi|x|F(\omega, q) - iryq]R(q)dS_q
$$

$$
= \int_{\mathbb{R}^2} \phi(q) \exp[i\pi|x|F(\omega, q) - iryq]R(q)|n(q)|dq_1dq_2
$$

$$
= (2\pi)|F''(\omega, q_c)|^{-1/2} \exp[i\pi \text{sgn}F''(\omega, q_c)/4]R(q_c)|n(q_c)|
$$

$$
\exp[i\pi|x|F(\omega, q_c) - irq_c]|rx|^{-1} + O(|rx|^{-2}).
$$

52
Since $H(z)f(x)$ in (5.3) has the two representations with $\Psi_{p,s}^+$ and $\Psi_{p,s}^-$, we may add them together to obtain $2H(z)f(x)$. It is convenient to do this because of the characteristic functions of the half spaces in $\Psi_{p,s}^\pm$. Thus, from (4.3), (5.3)

$$2v_S(x; z) = \Psi_s^+(c_s^2 |\eta|^2 - z)^{-1}\Psi_s^+ f(x) +$$
$$\Psi_s^-(c_s^2 |\eta|^2 - z)^{-1}\Psi_s^- f(x)$$
$$= \Psi_{sh}^+(c_s^2 |\eta|^2 - z)^{-1}\Psi_{sh}^+ f(x) +$$
$$\Psi_{sh}^-(c_s^2 |\eta|^2 - z)^{-1}\Psi_{sh}^- f(x) +$$
$$\Psi_{sv}^+(c_s^2 |\eta|^2 - z)^{-1}\Psi_{sv}^+ f(x) +$$
$$\Psi_{sv}^-(c_s^2 |\eta|^2 - z)^{-1}\Psi_{sv}^- f(x)$$

(7.3)

$$\equiv 2v_{sh}(x; z) + 2v_{sv}(x; z),$$

where we have used the fact that $h(q)$ is orthogonal to $v(q)$ (see (2.6), (4.4)).

Now let $\nu \in [\nu_0 - \delta], \nu_0 + \delta, \nu_0 - 4\delta > 0, \psi \in C_0^\infty(\mathbb{R}),$ supp $\psi \subset \{r : |c_s r - \nu_0| \leq 4d\} \subset (0, \infty)$, $\psi(r) = 1$ for $|c_s r - \nu_0| \leq 3\delta, \chi \in C_0^\infty(\mathbb{R}^3\setminus\{0\}), \chi(\eta) = \chi(rq) \equiv \psi(r), q \in S^2, z = (\nu \pm i\epsilon)^2, \epsilon \in (0, \epsilon_0]$. Setting $\Phi f(\eta) + \Phi f(\tilde{\eta}) = \Phi g(\eta)$, from (4.3), (5.3) after straightforward changes of the variable of integration we have

$$v_{sh}(x; z) = (2\pi)^{-3/2} \int (c_s^2 |\eta|^2 - z)^{-1} \exp(ix\eta)h(q) \otimes h(q)\Phi g(\eta) d\eta$$

53
\[ \equiv I_h(x; z) + J_h(x; z), \]

\[ J_h(x; z) = (2\pi)^{-3/2} \int \left( c^2 |\eta|^2 - z \right)^{-1} \left[ 1 - \chi(\eta) \right] \exp(ix\eta) \times \]

\[ h(q) \otimes h(q) \Phi g(\eta) d\eta \quad (7.4) \]

\[ = 0(|x|^{-3+\mu}), \quad \mu > 0, \]

\[ I_h(x; z) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left( c^2 |\eta|^2 - z \right)^{-1} \chi(\eta) \exp[i(x-y)] \times \]

\[ h(q) \otimes h(q) g(y) d\eta dy \]

\[ = (2\pi)^{-3} \int_{\mathbb{R}^3} D_h(x, y; z) g(y) dy, \]

\[ D_h(x, y; z) = \int_0^\infty \left( c^2 r^2 - z \right)^{-1} \psi(r) r^2 dr \int_{S^2} \exp[i(r(x-y)q)] h(q) \otimes h(q) dS, \]

where the estimate for \( J_h(x; z) \) follows from the fact that it can be viewed as a singular integral operator [12, 16].

The principle of stationary phase (7.1) applied to the integral over \( S^2 \) for \( x = |x|\omega, \omega \in S^2 \), in the expression for \( D_h \) gives [16, 19]

\[ \int_{S^2} \exp[i(r(x-y)q)] h(q) \otimes h(q) dS = (2\pi)|rx|^{-1} h(\omega) \otimes h(\omega) \times (7.5) \]

\[ \sum_{j=\pm 1} \exp(i\pi_t - ij\pi/2) + q(rx), \]

54
Applying (7.5) to $D_h$, we obtain

$$D_h(x, y; z) = (2\pi)|x|^{-1}h(\omega) \otimes h(\omega) \sum_{j=\pm 1} \exp(-ij\pi/2) \times$$

$$\int_0^\infty [r - (\nu \pm i\epsilon)]^{-1} \phi(r) \exp(itjr/c_s) dr$$

$$+ 0(|x|^{-\kappa - 1}), \kappa \in (1/2, 1) \text{ arbitrarily close}$$

to one, $\phi(r) = re_s^{-2}\psi(r/c_s)(r + \nu \pm i\epsilon)^{-1}$.

(The remainder estimate is obtained as in [11, 16].) Using now the Fourier transform in the usual way [16], we find that $D_h(x, y; \nu \pm i\epsilon)$ are bounded and continuous for $x, y \in \mathbb{R}^3, \nu \in [\nu_0 - \delta, \nu_0 + \delta], \epsilon \in [0, \epsilon_0]$, and for large $|x|

$$D_h(x, y; \nu \pm i0) = 2\pi^2 c_s^{-2}|x|^{-1}h(\omega) \otimes h(\omega) \exp[i\nu(\pm |x| + y\omega)/c_s] +$$

$$0(|x|^{-1-\kappa}) \quad (7.6)$$

Recalling now the definition of $g$ above and noting that $\Phi f(\tilde{\eta}) = \Phi \tilde{f}(\eta)$ where $\text{supp} \tilde{f} \subset \mathbb{R}_+^3$, $\tilde{f}(y) = f(\tilde{y}), \tilde{y} = (y', -y_3)$, from (7.4), (7.6) we have finally

$$v_{sh}(x; (\nu \pm i0)^2) = g_s^\pm(x; \nu) h(\omega) \otimes h(\omega) F_h(\pm \omega; \nu) + 0(|x|^{-1-\kappa}),$$

$$g_s^\pm(x; \nu) = (4\pi|x|)^{-1} \exp(\pm i\nu|x|/c_s),$$

55
\[ F_h(\omega; \nu) = c_s^{-2} \int \exp(-i\nu \omega y/c_s) [f(y) + \tilde{f}(y)] dy, \quad (7.7) \]

\[ x = |x|\omega, \ \omega \in S^2, \ \kappa \in (1/2, 1) \]

arbitrarily close to one, \( \nu \in \mathbb{R}_+ \).

Thus, in \( \mathbb{R}_+^3 \), as expected because of the lack of coupling, in order \( |x|^{-1} v_{sh}(x; (\nu \pm i0)^2) \exp(-i\nu t) \) are outgoing (+) and incoming (−) hemispherical SH waves. (Note that \( g_s^{\pm} \) are the Green functions for \( S \) waves; see §2.)

We proceed to \( v_{sv}(x; z) \). In the notation of (4.3) let

\[ \pi_s^{\pm}(x, \eta) = \exp(i\eta \nu) v(q) - \exp(i\eta \nu) r_s^{\pm}(q) v(q) \]

\[ - \exp(i\eta \nu) r_s^{\pm}(q) \theta(q). \quad (7.8) \]

Then from (4.3), (5.3), (7.3)

\[ v_{sv}(x; z) = 2^{-1}[(\Psi_{sv}^+)^*(c_s^2| \cdot |^2 - z)^{-1} \Psi_{sv}^+ f(x) + \]

\[ (\Psi_{sv}^-)^*(c_s^2| \cdot |^2 - z)^{-1} \Psi_{sv}^- f(x)] \]

\[ \equiv v_{sv}^1(x; z) + v_{sv}^2(x; z), \quad (7.9) \]

\[ v_{sv}^1(x; z) \equiv v_{sv}^1(x; z; \psi) = 2^{-1}(2\pi)^{-3/2} \int \int (c_s^2 r^2 - z)^{-1} \psi(r) \times \]

\[ \{ \chi_-(q) \pi_s^+(x, \eta) \otimes [v(q) \Phi f(\eta) - r_s^+(q) v(q) \Phi f(\tilde{\eta}) \]

\[ - r_p^+(q) \theta^+(q) \Phi f(\tilde{\eta}_s^+)] \]
\[ + \chi(q)\pi^{-}(x, \eta) \otimes [v(q)\Phi f(\eta) - r^{-}_{ss}(q)v(q)\Phi f(\bar{\eta})]
\]

\[ - r^{-}_{ps}(q)\bar{\theta}^{-}(q)\Phi f(\bar{\eta}^{-})]d\eta. \]

\[ v_{sv}^{2}(x; z) \equiv v_{sv}^{1}(x; z; 1 - \psi). \]

Replacing the interval \(|c_{s}r - \nu_{0}| \leq 2\delta\) by a circle of radius \(2\delta < \epsilon_{0}\) in the lower \((z = (\nu + i\epsilon)^{2})\) or upper \((z = (\nu - i\epsilon)^{2})\) half plane and extending \(\psi\) to this disk by 1, it follows that \(v_{sv}^{1}(x; (\nu \pm i0)^{2})\) exists and is continuous in \((x, \nu), \nu \in [\nu_{0} - \delta, \nu_{0} + \delta]\). Hence, \(v_{sv}^{1}(x; (\nu \pm i\epsilon)^{2})\) exists and is continuous on \(\mathbb{R}^{3}_{+} \times [\nu_{0} - \delta, \nu_{0} + \delta] \times [0, \epsilon_{0}]\). From the estimates below it follows that \(v_{sv}^{2}\) is also bounded. Since the integrand of \(v_{sv}^{2}\) contains no singularity at \(c_{s}r = \nu \pm i0\), it is also bounded and continuous. In summary, \(v_{sv}(x; z)\) is bounded and continuous on \(\mathbb{R}^{3}_{+} \times [\nu_{0} - \delta, \nu_{0} + \delta] \times [0, \epsilon_{0}]\). The same obviously applies to \(D^{\delta}v_{sv}(x; z)\) i.e., the principle of limiting absorption holds.

To study the asymptotics of \(v_{sv}(x; (\nu \pm i0)^{2})\), we write (7.9) as

\[ v_{sv}(x; z) = k^{s}(x; z) - \ell^{s}(x; z) - m^{s}(x; z), \quad (7.10) \]

where

\[ k^{s}(x; z) = \]

\[ = 2^{-1}(2\pi)^{-3/2} \int (c_{3}^{2}|\eta|^{2} - z)^{-1}[\chi_{-}(q_{3})\pi^{+}_{s}(x, \eta) + \chi_{+}(q_{3})\pi^{-}_{s}(x, \eta)] \otimes v(q)\Phi f(\eta)d\eta \]

57
\[ = 2^{-1}(2\pi)^{-3/2} \int (c_s^2 |\eta|^2 - z)^{-1} \{ \exp(i x \eta) v(q) - \exp(i x \bar{\eta}) \} \]

\[ [\chi_-(q_3) r_{ss}^+(q) + \chi_+(q_3) r_{ss}^-(q)] v(q) - [\chi_-(q_3) \exp(i x \eta^s_3) \]

\[ r_{ps}^+(q) \theta^+(q) + \chi_+(q_3) \exp(i x \eta^-_s) r_{ps}^-(q) \theta^-(q) ] \times \]

\[ v(q) \Phi f(\eta) d\eta, \]

\[ \equiv k_1^s(x; z) + k_2^s(x; z) + k_3^s(x; z), \]

\[ \ell(x; z) = \]

\[ = 2^{-1}(2\pi)^{-3/2} \int (c_s^2 |\eta|^2 - z)^{-1} [\chi_-(q_3) \pi^+(x, \eta) r_{ss}^+(q) + \]

\[ \chi_-(q_3) r_{ss}^-(q)] \otimes v(q) \Phi f(\bar{\eta}) d\eta \] (7.11)

\[ = 2^{-1}(2\pi)^{-3/2} \int (c_s^2 |\eta|^2 - z)^{-1} \{ \exp(ix\eta)[\chi_-(q_3) r_{ss}^-(q) + \]

\[ \chi_+(q_3) r_{ss}^-(q)] v(q) - \exp(i x \bar{\eta}) [\chi_-(q_3) r_{ss}^+(q) r_{ss}^+(q) + \chi_+(q_3) \]

\[ r_{ss}^+(q)^2] v(q) - [\chi_-(q_3) \exp(i x \eta^s_3) r_{ps}^+(q) r_{ss}^+(q) \theta^+(q) + \]

\[ \chi_+(q_3) \exp(i x \eta^-_s) r_{ps}^-(q) r_{ss}^-(q) \theta^-(q) ] \} \otimes v(q) \Phi f(\bar{\eta}) d\eta, \]

\[ \equiv \ell_1(x; z) + \ell_2(x; z) + \ell_2(x; z), \]

\[ m(x; z) = \]

\[ = 2^{-1}(2\pi)^{-3/2} \int (c_s^2 |\eta|^2 - z)^{-1} \{ \chi_-(q_3) \pi^+(x, \eta) \otimes r_{ps}^+(q) \theta^+(q) \]
\[
\Phi f(\eta_s^+) + \chi_+(q_3) \pi^- (x, \eta) \otimes r_{ps}^- (q) \theta^- (q) \Phi f(\eta_s^-) \bigg\} d\eta
\]

\[
= 2^{-1}(2\pi)^{-3/2} \int (c_s^2 |\eta|^2 - z)^{-1} \{ \exp(i x \eta) [\chi_-(q_3) r_{ps}^+(q) v(q) \otimes \\
\theta^+(q) \Phi f(\eta_s^+) + \chi_+(q_3) r_{ss}^-(q) r_{ps}^- (q) v(\bar{q}) \otimes \theta^-(q) \Phi f(\eta_s^-)] - \\
\exp(i x \bar{\eta}) [\chi_-(q_3) r_{ss}^+(q) r_{ps}^+(q) v(\bar{q}) \otimes \theta^+(q) \Phi f(\eta_s^+) + \\
\chi_+(q_3) r_{ss}^-(q) r_{ps}^- (q) v(\bar{q}) \otimes \theta^-(q) \Phi f(\eta_s^-)] - \\
[\chi_-(q_3) \exp(i x \eta_s^+) (r_{ps}^+(q))^2 \theta^+(q) \otimes \theta^+(q) \Phi f(\eta_s^+) + \\
\chi_+(q_3) \exp(i x \eta_s^-) (r_{ps}^- (q))^2 \theta^-(q) \otimes \theta^-(q) \Phi f(\eta_s^-)] \} d\eta
\]

\[
\equiv m_1(x; z) + m_2(x; z) + m_3(x; z).
\]

We first consider \( k^s \) in all detail, treating \( k_{11}^s, k_{12}^s, k_{33}^s \) successively. The arguments for \( \ell^s \) and \( m^s \) then follow one of the patterns established, and so we state only the final result. Introducing the function \( \chi \) as above, we write

\[
k_{11}^s(x; z) = k_{11}^s(x; z) + k_{12}^s(x; z),
\]

\[
k_{12}^s(x; z) = 2^{-1}(2\pi)^{-3/2} \int (c_s^2 |\eta|^2 - z)^{-1} [1 - \chi(\eta)] \\
\exp(i x \eta) v(q) \otimes v(q) \Phi f(\eta) d\eta
\]

\[
= 0(|x|^{-3+\mu}), \quad \mu > 0,
\]

59
\[ k_{11}^s(x; z) = 2^{-1}(2\pi)^{-3/2} \int (c_s^2|\eta|^2 - z)^{-1} \chi(\eta) v(q) \otimes v(q) \Phi f(\eta) d\eta \]

\[ = 2^{-1}(2\pi)^{-3} \int K_{11}(x, y; z) f(y) dy, \quad (7.12) \]

\[ K_{11}(x, y; z) = \int_0^\infty (c_s^2r^2 - z)^{-1} \psi(r)r^2 dr \int_{S^2} \exp[ir(x - y)] v(q) \otimes v(q) dS_q. \]

Applying (7.1) to the integral over \( S^2 \) and then using the Fourier transform in the usual way [16], for \( x = |x|\omega, |x| \text{ large} \)

\[ K_{11}(x, y; (\nu \pm i0)^2) = 2\pi c_s^2 |x|^{-1} v(\omega) \otimes v(\omega) \]

\[ \exp[i\nu(\pm |x| \mp y\omega)/c_s] + \]

\[ 0(|x|^{-1-\kappa}), \quad \kappa \in (1/2, 1) \quad (7.13) \]

arbitrarily close to one.

From (7.10), (7.12), (7.13)

\[ k_1^s(x; (\nu \pm i0)^2) = g_1^s(x; \nu) v(\omega) \otimes v(\omega) K_1^s(\pm \nu; v) + 0(|x|^{-1-\kappa}), \]

\[ K_1^s(\omega; \nu) = 2^{-1}c_s^{-2} \int \exp(-i\nu\omega y/c_s) f(y) dy, \quad (7.14) \]

\[ \kappa \in (1/2, 1) \text{ arbitrarily close to one.} \]

We now proceed to \( k_2^s \). With the function \( \chi \) above

\[ -k_2^s(x; z) = k_2^{s1}(x; z) + k_2^{s2}(x; z), \]

60
\[ k_{21}^s(x; z) = 2^{-1}(2\pi)^{-3/2} \int (c_s^2|\eta|^2 - z)^{-1} \exp(ix\tilde{\eta})\chi(\eta)[\chi_-(q_3)r_{ss}^+(q) + \chi_+(q_3)r_{ss}^-(q)]v(q) \otimes v(q)\Phi f(\eta)d\eta, \]

(7.15)

\[ k_{22}^s(x; z) = 2^{-1}(2\pi)^{-3/2} \int \exp(ix\tilde{\eta})(c_s^2|\eta|^2 - z)^{-1}[1 - \chi(\eta)] \]

\[ [\chi_-(q_3)r_{ss}^+(q) + \chi_+(q_3)r_{ss}^-(q)]v(q) \otimes v(q)\Phi f(\eta)d\eta \]

\[ \equiv \int \exp(ix\tilde{\eta})[g^+(\eta; z) + g^-(\eta; z)]d\eta \]

in an obvious notation; we have

\[ g^+(\xi, 0; z) = g^-(\xi, 0; z), \]

since \( \tilde{\Delta}^+/\Delta^+ = \tilde{\Delta}^-/\Delta^- \) for \( q_3 = 0 \) (see (4.3)).

For \( x = |x|\omega \in \mathbb{R}^3_+; \tilde{\omega} = (\omega', -\omega_3) \) let \( L = \tilde{\omega} \cdot \nabla_\eta; \) then for a scalar function \( g \) \( L^*g(\eta) = -\nabla_\eta[g(\eta)\tilde{\omega}] \). With \( \eta = (\xi, \rho) \in \mathbb{R}^3 \) to simplify notation we drop the vector indices (writing, for example, \( g \) in place of \( g_3 \)) and compute

\[ k_{22}^s(x; z) = \]

\[ = (i|x|)^{-1} \left\{ \begin{array}{c}
\int_{\mathbb{R}^3}_- [L \exp(ix\tilde{\eta})]g^+(\eta; z)d\eta + \\
\int_{\mathbb{R}^3}_+ [L \exp(ix\tilde{\eta})]g^-(\eta; z)d\eta
\end{array} \right\} \]
\[= \omega_3 (i|x|)^{-1} \left\{ \int_{\mathbb{R}^2} \exp(ix^t \xi) [g^+(\xi, 0^-; z) - g^-(\xi, 0^+; z)] d\xi \right\} + \]

\[(i|x|)^{-1} \left\{ \int_{\mathbb{R}^3_-} \exp(ix\tilde{\eta}) L^* g^+(\eta; z) + \int_{\mathbb{R}^3_+} \exp(ix\tilde{\eta}) L^* g^-(\eta; z) \right\} \]

\[= -|x|^{-2} \left\{ \int_{\mathbb{R}^3_-} [L \exp(ix\tilde{\eta})] L^* g^+(\eta; z) d\eta + \int_{\mathbb{R}^3_+} [L \exp(ix\tilde{\eta})] L^* g^-(\eta; z) d\eta \right\} \]

\[= \omega_3 |x|^{-2} \left\{ \int_{\mathbb{R}^2} \exp(ix^t \xi) [L^* g^+(\xi; 0; x) - L^* g^-(\xi; 0; z)] d\xi \right\} \]

\[= \omega_3 |x|^{-2} \left\{ \int_{\mathbb{R}^3_-} \exp(ix\tilde{\eta}) (L^*)^2 g^+(\eta; z) d\eta + \int_{\mathbb{R}^3_+} \exp(ix\tilde{\eta}) (L^*)^2 g^-(x; z) d\eta \right\} . \]

Here the integrand in the integral over \(\mathbb{R}^2\) is at worst \(O(|\xi|^{-1})\) near 0, while that of the integrals over \(\mathbb{R}^3_\pm\) are at worst
$0(|\eta|^{-2})$ near 0. These integrals are thus bounded by $\text{const}(f)$, and $k_{22}(x; z)$ is $0(|x|^{-2})$.

From (7.15)

$$k^s_{21}(x; z) = 2^{-1}(2\pi)^{-3/2} \int_{\mathbb{R}^3} K_{21}(x, y; z) f(y) dy,$$

(7.17)

$$K_{21}(x, y; z) = \int_0^{\infty} (c^2 r^2 - z)^{-1} \psi(r) r^2 dr \int_{S^2} \exp(i r |x| \tilde{\omega} q - iyr q)$$

$$\left[ \chi_-(q_3) r^+_{ss}(q) + \chi_+(q_3) r^-_{ss}(q) \right] v(\tilde{q}) \otimes v(q) dS_q,$$

$$\tilde{\omega} = (\omega', -\omega_3), \quad \omega_3 > 0.$$

Now let $\phi_+$ and $\phi_-$ in $C_0^\infty(S^2)$ be equal to one in small neighborhoods of the critical points $\tilde{\omega}$ and $-\tilde{\omega}$ respectively, whereby $\text{supp} \phi^\pm \cap \{ q_3 = 0 \} = \emptyset$, and set $\bar{\phi} = 1 - \phi^+ - \phi^-$. Identifying $\pm$ and $\pm 1$, we have

$$K_{21}(x, y; z) = K^+_{21}(x, y; z) + K^-_{21}(x, y; z) + \tilde{K}_{21}(x, y; z),$$

$$K^j_{21}(x, y; z) = \int_0^{\infty} (c^2 r^2 - z)^{-1} P(j, r, x, y) dr,$$

$$P^j(r, x, y) = \int_{S^2} \phi^j(q) \exp(i r |x| \tilde{\omega} q - iyr q) r^j_{ss}(q) v(\tilde{q}) \otimes v(q) dS_q$$

(7.18)
\[ \tilde{K}_{21}(x, y; z) = \int_0^\infty (c_s^2 r^2 - z)^{-1} \psi(r) r^2 \tilde{I}(r, x, y) dr, \]

\[ \tilde{I}(r, x, y) = \int_{S^2} \tilde{\phi}(q) \exp(ir|x|\tilde{\omega} q - i r y q) [\chi_-(q_3) r_{ss}^-(q) + \chi_+(q_3) r_{ss}^+(q)] v(\tilde{q}) \otimes v(q) dS. \]

Setting \( t_j = j|x|-i j\tilde{\omega} y, j = \pm 1, \) and applying (7.1), we have

\[ I^j(r, x, y) = -i j^2 \pi r_{ss}^j(j\tilde{\omega}) \exp(i r t_j) v(\omega) \otimes v(\tilde{\omega}) |r x|^{-1} + 0(|r x|^2), \]

so that from (7.16)

\[ K_{21}^j(x, y; z) = -i j^2 \pi r_{ss}^j(j\tilde{\omega}) v(\omega) \otimes v(\tilde{\omega}) |x|^{-1} \int_0^\infty [r - (\nu \pm i \epsilon)]^{-1} \lambda(r) \exp(i r t_j/c_s) dr, \] (7.19)

\[ \lambda(r) = r c_s^{-2} \psi(r/c_s) (r + \nu \pm i \epsilon)^{-2}. \]

From this it follows in the usual way [16] that

\[ K_{21}^j(x, y; (\nu + i j 0)^2) = 2 \pi^2 r_{ss}^j(\omega) v(\omega) \otimes v(\tilde{\omega}) |x|^{-1} \exp[i\nu(j|x| - j\tilde{\omega} y)/c_s] + 0(|x|^{-1-\kappa}), \] (7.20)

\[ K_{21}^j(x, y; (\nu - i j 0)^2) = 0(|x|^{-2}), j = \pm 1, \ k \in (1/2, 1). \]

By localizing in a neighborhood of \( \{q_3 = 0\} \) and introducing polar coordinates, it can be shown that in (7.18) \( \tilde{I}(r, x, y) = \)
0(|x|^{-2}), \text{ whence } K_{21}(x, y; z) = 0(|x|^{-1-\kappa}), \ \kappa \in (1/2, 1), \text{ uniformly with respect to } y \text{ in compact sets, } \nu \in [\nu_0 - \delta, \nu_0 + \delta], \text{ and } \epsilon \in (0, \epsilon_0] \text{ (see the arguments following (8.11) below).}

From (7.15), (7.16), (7.17), (7.20) we finally obtain for \( x = |x| \omega, |x| \text{ large,} \)
\[
k_s^*(x; (\nu \pm i0)^2) = -g_s^\pm(x; \nu) r_{ss}^\pm(\omega) v(\omega) \otimes v(\tilde{\omega}) K_2^*(\pm \tilde{\omega}; \nu) + 0(|x|^{-1-\kappa}),
\]
\[
K_2^* = K_1^* \text{ of (6.14), } \kappa \in (1/2, 1).
\tag{7.21}
\]

We now consider the term \( k_3^s \) of (7.12):

\[
-k_3^s(x; z) = k_{31}^s(x; z) + k_{32}^s(x; z),
\]

\[
k_{31}^s(x; z) = \int (c_s^2 |\eta|^2 - z)^{-1} \chi(\eta)[\chi_-(q_3) \exp(ix\eta_+^s q_1^s \theta^+(q) + \chi_+(q_3) \exp(ix\eta_-^s q_1^s \theta^-(q)] v(q) \Phi f(\eta) d\eta,
\tag{7.22}
\]

\[
k_{32}^s(x; z) = \int (c_s^2 |\eta|^2 - z)^{-1}[1 - \chi(\eta)] [\chi_-(q_3) \exp(ix\eta_+^s q_1^s \theta^+(q) + \chi_+(q_3) \exp(ix\eta_-^s q_1^s \theta^-(q)] v(q) \Phi f(\eta) d\eta
\]

\[
\equiv \int_{\mathbb{R}^3} \exp(ix\eta_+^s g^+(\eta; z) d\eta + \int_{\mathbb{R}^3} \exp(ix\eta_-^s g^-(\eta; z) d\eta,
\]

\[
0 = g^+(\xi, 0^-; z) = g^-(\xi, 0^+; z),
\]
where the last line follows from \( r_{ps}^\pm(q) = 0 \) for \( q_3 = 0^\mp \) (see (4.3)).
We proceed to estimate \( k_{32}^3(x; z) \) of (7.22). From (4.3) for \( q = (q', q_3) \in S^2 \)
\[
\omega \cdot \eta_s = \begin{cases} \omega' \eta' \pm \sqrt{[\eta'^2 - (n^2 - 1)|\eta'|^2]}, |q'| < n, \\
\omega' \eta' + i\omega_3 \sqrt{[(n^2 - 1)|\eta'|^2 - \eta_3^2]}, |q'| > n \end{cases}.
\]

In the integral (7.22) we make the change of variable \( \eta' = \tilde{n} \tilde{\eta}' \), \( \eta_3 = \tilde{n}_3, \tilde{n} = (n^2 - 1)^{-1/2}, d\eta = \tilde{n}^2 d\tilde{\eta} \), and we then retain the previous notation with \( \eta \) in place of \( \tilde{\eta} \). Then
\[
\omega \cdot \eta_s = \begin{cases} \ell_\pm(\omega, \eta) = \tilde{n} \omega' \eta' \pm \omega_3 \sqrt{[\eta_3^2 - |\eta'|^2]}, |\eta'| < |\eta_3|, \\
\ell(\omega, \eta) = \tilde{n} \omega' \eta' + i\omega_3 \sqrt{[|\eta'|^2 - \eta_3^2]}, |\eta'| > |\eta_3| \end{cases}.
\]

Integration by parts is permissible in the interior (\( |\eta'| < |\eta_3| \)) and exterior (\( |\eta'| > |\eta_3| \)) of the cones \( C_\pm = \{ \pm \eta_3 = \eta' \} = \{ \eta_\pm(s, t) \in \mathbb{R}^3 : \eta_\pm(s, t) = (t/\sqrt{2})(\cos s, \sin s, \pm 1), s \in (0, 2\pi], t > 0 \} \) with inner unit normals \( \nu^\pm = (1/\sqrt{2})(-\cos s, -\sin s, \pm 1) \) and element of surface area \( dS = (1/\sqrt{2})tdsdt \). We denote the interior of the cones \( C_\pm \) by \( [C_\pm^-] \) the exterior by \( [C_\pm^+] \), and define for a scalar function \( g \)
\[
L_> = |\nabla \ell(\omega, \eta)|^{-2} \nabla \bar{\ell}(\omega, \eta) \cdot \nabla,
\]
\[
L_\pm^\pm = |\nabla \ell_\pm(\omega, \eta)|^{-2} \nabla \ell_\pm(\omega, \eta) \cdot \nabla,
\]
\[
L^*_g = -\nabla \cdot [\nabla \bar{\ell}(\omega, \eta)g/|\nabla \ell(\omega, \eta)|^2],
\]
\[
(L_\pm^\pm)^* g = -\nabla \cdot [\nabla \ell_\pm(\omega, \eta)g/|\nabla \ell_\pm(\omega, \eta)|^2].
\]
Since $\nu^\pm \cdot \nabla \ell /|\nabla \ell|^2$ and $\nu^\pm \cdot \nabla \ell^\pm /|\nabla \ell^\pm|^2$ are zero on the cones $C_\pm$ and $\partial_3 \ell(\omega, \eta', 0) = 0$ (see (7.27), (7.28) below), dropping the vector indices on $g$, etc., we have by integration by parts in (7.22)

\[
ix|k_{32}^\pm(x; z) = \int_{|C^>_\pm|} [L_> \exp(ix\eta^+_s)]g^+(\eta; z)d\eta + \\
\int_{|C^<_\pm|} [L^>_\pm \exp(ix\eta^+_s)]g^+(\eta; z)d\eta + \\
\int_{|C^<_\pm|} [L_> \exp(ix\eta^-_s)]g^-(\eta; z)d\eta + (7.25) \\
\int_{|C^>_\pm|} [L^<_\pm \exp(ix\eta^+_s)]g^-(-\eta; z)d\eta \\
\int_{|C^>_\pm|} \exp(ix\eta^+_s)L^*_\pm g^+(\eta; z)d\eta + \\
= \int_{|C^<_\pm|} \exp(ix\eta^-_s)(L^<_\pm)^* g^+(\eta; z)d\eta + \\
\int_{|C^>_\pm|} \exp(ix\eta^-_s)L^*_\pm g^-(\eta; z)d\eta + \\
\int_{|C^>_\pm|} \exp(ix\eta^-_s)L^*_\pm g^-(\eta; z)d\eta + 
\]

67
\[
\int_{|C^<_\pm|} \exp(ix\eta^-)(L^-)^* g^- (\eta, z)d\eta.
\]

A further integration by parts introduces a singularity on the cones \(C_{\pm}\) from either side of them. However, Providence has designed that the singularities occur with null coefficients. Some care is needed to reveal this design. To this end we introduce new coordinates based on the cones \(C_{\pm}\): for \(0 < t < \infty, \ s \in (0, 2\pi), \ u \in (0, t)\)

in \([C^<_\pm]\): \(\eta = \eta(s, t, u) = \eta_{\pm}(s, t) + u\nu^\pm\)

\[\begin{align*}
&= (1/\sqrt{2})((t - u) \cos s, (t - u) \sin s, \pm t \pm u), \\
&|\partial(\eta_1, \eta_2, \eta_3)/\partial(s, t, u)| = (t - u)/\sqrt{2}, \\
&\eta^2_3 - |\eta'|^2 = 2ut, \ |\eta|^2 = t^2 + u^2
\end{align*}\] (7.26)

in \([C^>_\pm]\): \(\eta = \eta(s, t, u) = \eta_{\pm}(s, t) - u\nu^\pm\)

\[\begin{align*}
&= (1/\sqrt{2})((t + u) \cos s, (t + u) \sin s, \pm t \mp u), \\
&|\partial(\eta_1, \eta_2, \eta_3)/\partial(x, t, u)| = (t + u)/\sqrt{2}, \\
&|\eta'|^2 - \eta^2_3 = 2ut, \ |\eta|^2 = t^2 + u^2.
\end{align*}\]

From (7.23), (7.24) with \(\eta = \eta(s, t, u)\) in the corresponding regions we compute

in \([C^>_\pm]\): \(\nabla \bar{\ell}(\omega, \eta) = (2ut)^{-1/2} t(\bar{n}\omega_1(2ut)^{1/2} - i\omega_3\eta_1, \)
\( \tilde{n}\omega_2(2ut)^{1/2} - i\omega_3\eta_2, i\omega_3\eta_3) \)

\[ = \tilde{n}'(\omega_1, \omega_2, 0) + (2ut)^{-1/2}i\omega_3'(-\eta_1, -\eta_2, \eta_3) \]

\[ \nabla \bar{\ell}(\omega, \eta)/|\nabla \ell(\omega, \eta)|^2 = 2ut \nabla \bar{\ell}(\omega, \eta)M(\omega, \eta), \]

\[ M(\omega, \eta) = [2\tilde{n}|\omega'|^2ut + \omega_3^2(t^2 + u^2)]^{-1}, \]

\[ \nabla \bar{\ell}(\omega, \eta) = -i\omega_3/(2ut)^{1/2} + i\omega_3(t^2 + u^2)/(2ut)^{3/2}, \]

\[ \nabla \bar{\ell}(w, \eta)/|\nabla \ell(\omega, \eta)|^2 = ia_1(\omega, \eta)(2ut)^{-1/2}, \]

\[ 0 < a_1(\omega, \eta) = \omega_3(t - u)^2M(\omega, \eta), \]

\[ \lim_{u \to 0} a_1(\omega, \eta) = \omega_3^{-1}, \]

\[ \nabla \bar{\ell} \cdot \nabla |\nabla \ell|^{-2} = ia_2(\omega, \eta)(2ut)^{-1/2} + a_3(u, t), \quad (7.27) \]

\[ a_2(\omega, \eta) = 4\omega_3ut[n^2|\omega'|^2(t^2 + u^2) + 2\omega_3^2ut]M(\omega, \eta) - \]

\[ 2\omega_3^2(t^2 + u^2)M(\omega, \eta), \]

\[ \lim_{u \to 0} a_2(\omega, \eta) = -2\omega_3^{-1}, \]

\[ a_3(\omega, \eta) = 2\tilde{n}[M - 2ut(\tilde{n}|\omega'|^2 + \omega_3^2)M^2]|\omega'|\eta', \]

\[ L^* g_\pm = -g_\pm[(\nabla \bar{\ell}/|\nabla \ell|^2) + \nabla \bar{\ell} \cdot \nabla |\nabla \ell|^{-2}] - (\nabla \bar{\ell} \cdot \nabla g_\pm)/|\nabla \ell|^2 \]

\[ = -g_\pm[ia(\omega, \eta)(2ut)^{-1/2} + a_3(\omega, \eta)] - 2ut M(\omega, \eta) \nabla \bar{\ell} \cdot \nabla g_\pm, \]
\[ a(\omega, \eta) = a_1(\omega, \eta) + a_2(\omega, \eta), \quad \lim_{u \to 0} a(\omega, \eta) = -\omega_3^{-1}. \]

In \([C^1] \): \[
\nabla \ell_\pm(\omega, \eta) = \bar{n}^\prime(\omega_1, \omega_2, 0) \pm (2ut)^{-1/2}\omega_3^\prime(-\eta_1, -\eta_2, \eta_3),
\]
\[
\begin{align*}
\nabla \ell_\pm(\omega, \eta)/|\nabla \ell_\pm(\omega, \eta)|^2 &= 2ut \nabla \ell_\pm(\omega, \eta) M_\pm(\omega, \eta), \\
M_\pm(\omega, \eta) &= [|\bar{n}(2ut)^{1/2}\omega' \mp \omega_3\eta'|^2 + \omega_3^2\eta_3^2]^{-1}, \\
\Delta \ell_\pm(\omega, \eta) &= \mp \omega_3(2ut)^{-1/2} \mp \omega_3(t^2 + u^2)(2ut)^{-3/2}, \\
\Delta \ell_\pm(\omega, \eta)/|\nabla \ell_\pm(\omega, \eta)|^2 &= \mp b_1^\pm(\omega, \eta)(2ut)^{-1/2}, \\
0 < b_1^\pm(\omega, \eta) &= \omega_3(t + u)^2 M_\pm(\omega, \eta), \\
\lim_{u \to 0} b_1^\pm(\omega, \eta) &= \omega_3^{-1}, \\
\nabla \ell_\pm \cdot \nabla |\nabla \ell_\pm|^{-2} &= \pm b_2^\pm(\omega, \eta)(2ut)^{-1/2} \pm b_3^\pm(\omega, \eta), \\
b_2^\pm, b_3^\pm &\in C^1, \\
\lim_{u \to 0} b_2^\pm &= 2\omega_3^{-1}, \\
b_2^\pm(\omega, \eta) &= 2\omega_3\{(t^2 + u^2)M_\pm(\omega, \eta) \mp 2utM_\pm^2(\omega, \eta) \\
&\quad [(\bar{n}(2ut)^{1/2}\omega' \mp \omega_3\eta')\eta' + \omega_3^2\eta_3^2]\}, \\
b_3^\pm(\omega, \eta) &= 2\bar{n}M_\pm(\omega, \eta)[2ut\omega_3(\bar{n}(2ut)^{1/2}|\omega'|^2 \mp \omega_3\omega'\eta')] \\
M_\pm(\omega, \eta) &= \omega'\eta'.
\]
\[(L^\pm)^* g_\pm = -g_\pm [\Delta \ell_\pm/|\nabla \ell_\pm|^2 + \nabla \ell_\pm \cdot \nabla |\nabla \ell_\pm|^{-2}] -
\]
\[
(\nabla \ell_\pm \cdot \nabla g_\pm)/|\nabla \ell_\pm|^2
\]
\[
= -g_\pm [\pm b_\pm^\pm(\omega, \eta)(2ut)^{-1/2} + b_3^\pm(\omega, \eta)] -
\]
\[2utM_\pm(\omega, \eta) \nabla \ell_\pm \cdot \nabla g_\pm,
\]
\[b^\pm(\omega, \eta) = b_2^\pm(\omega, \eta) - b_1^\pm(\omega, \eta) \in C^1,
\]
\[\lim_{u \to 0} b^\pm(\omega, \eta) = \omega_3^{-1}.
\]

In (7.23) we now replace the cones \(C_\pm\) by \(C^\pm_\pm(u_0) = C_\pm + u_0 \nu^\pm\) and \(C^\pm_\pm(u_0) = C_\pm - u_0 \nu^\pm\) and denote their interiors by \([C^\pm_\pm(u_0)]\) and their exteriors by \([C^\pm_\pm(u_0)]\), \(u_0 \geq 0\) small. Omitting the explicit dependence of \(\eta\) on \(s, t, u\), we have
\[-|x|^2 \lambda^*_3(x; z) =
\]
\[
= \lim_{u_0 \to 0} \begin{cases}
\int_{[C^\pm_\pm(u_0)]} L_\gamma \exp(ix\eta_3^\pm) L_\gamma^* g^\pm(\eta; z) d\eta +
\int_{[C^\pm_\pm(u_0)]} (L^\pm_\gamma)^* \exp(ix\eta_3^\pm)(L^\pm_\gamma)^* g^\pm(\eta; z) d\eta +
\int_{[C^\pm_\pm(u_0)]} L_\gamma \exp(ix\eta_3^-) L_\gamma^* g^-(\eta; z) d\eta +
\int_{[C^\pm_\pm(u_0)]} L_\gamma \exp(ix\eta_3^+) L_\gamma^* g^+(\eta; z) d\eta +
\end{cases}
\]
\[
\int_{[C^<_u(u_0)]} (L^-) \exp(ix\eta_s^-)(L^-)^* g^- (\eta; z) d\eta
\]

\[= \lim_{u_0 \to 0} \left\{ \int_{C^-_u(u_0)} - (\nu^- \cdot \nabla \ell_/-|\nabla \ell_-|^2) \exp(ix\eta_s^+) (L^+_\ell)^* g^+ (\eta; z) dS \right.\]

\[\int_{C^-_u(u_0)} (\nu^- \cdot \nabla \ell_-/|\nabla \ell_-|^2) \exp(ix\eta_s^-) (L^-_\ell)^* g^- (\eta; z) dS \]

\[\int_{C^+_u(u_0)} - (\nu^+ \cdot \nabla \ell_+/|\nabla \ell_+|^2) \exp(ix\eta_s^-) (L^-_\ell)^* g^- (\eta; z) dS \]

\[\int_{C^+_u(u_0)} (\nu^+ \cdot \nabla \ell_+/|\nabla \ell_+|^2) \exp(ix\eta_s^-) (L^-_\ell)^* g^- (\eta; z) dS \right\} \]

\[+ \lim_{u_0 \to 0} \left\{ \int_{[C^>_u(u_0)]} \exp(ix\eta_s^+) (L^+_\ell)^2 g^+ (\eta; z) d\eta + \right.\]

\[\int_{[C^>_u(u_0)]} \exp(ix\eta_s^+) [(L^+_\ell)^*]^2 g^+ (\eta; z) d\eta + \right.\]

\[\int_{[C^>_u(u_0)]} \exp(ix\eta_s^-) (L^+_\ell)^2 g^- (\eta; z) d\eta + \]
\[
\int_{[C^<_+(u_0)]} \exp(ix\eta^-) \left[ (L^-)_+ \right]^2 g^- (\eta; z) \, d\eta.
\]

The first limit exists and in modulus is \( \leq c(f) \), since, e.g., \( \nu^- \cdot \nabla \ell / |\nabla \ell|^2 \) is proportional to \((2ut)^{1/2}\), while \( L^*_+ g^+ \) is proportional to \((2ut)^{-1/2}\) (see (7.27)); there is thus no singularity on the cone \( C^>_+ \), and the surface integral is \( \leq c(f) < \infty \) in modulus. Since the left side of (7.29) does not depend on \( u \) and the first limit is \( \leq c(f) < \infty \), it follows that the second limit is \( = F(x), |F(x)| < \infty \). Our task is to show that in fact \( |F(x)| \leq c(f) < \infty \). This is true, since the singularity of the term \((L^+_*)^2 g \propto (2ut)^{-1}\) has a coefficient which vanishes at \( u = 0 \), as we now proceed to demonstrate.

We consider, e.g., \([C^>_+(u_0)]\) and \([C^<_+(u_0)]\); the other two regions can be treated in the same way. From (7.27), (7.28) for \( \eta \in [C^>_+(u_0)], [C^<_+(u_0)] \), respectively,

\[
[L^*_+]^2 g^- =
\]

\[
= -L^*_+ g^- \left[ ia(\omega, \eta)(2ut)^{-1/2} + a_3(\omega, \eta) \right] - 2ut M(\omega, \eta) \nabla \ell^- \cdot \nabla L^*_+ g^- \\
= g^- \left[ ia(\omega, \eta)(2ut)^{-1/2} + a_3(\omega, \eta) \right]^2 + 2ut M(\omega, \eta) \nabla \ell \cdot \nabla g^- \\
\quad \left\{ [ia(\omega, \eta)(2ut)^{-1/2} + a_3(\omega, \eta)] g^- + 2ut M(\omega, \eta) \nabla \ell \cdot \nabla g^- \right\} \quad (7.30)
\]

\[
= -g^- (\eta; z) a(\omega, \eta)^2 (2ut)^{-1} + 2iuta(\omega, \eta) M(\omega, \eta)
\]

73
\[ g^{-} \nabla \bar{\ell} \cdot \nabla (2ut)^{-1/2} + (2ut)^{1/2} h^{-} (\omega, \eta) + k^{-} (\omega, \eta) \]

\[ = -g^{-} (\omega, \eta) a (\omega, \eta) [a (\omega, \eta) + \omega_{3} M (\omega, \eta) (t^{2} + u^{2})] (2ut)^{-1} - \]

\[ \tilde{g}^{-} (\eta; z) a (\omega, \eta) M (\omega, \eta) \omega' \eta' (2ut)^{-1/2} + h^{-} (\omega, \eta) (2ut)^{1/2} + \]

\[ k^{-} (\omega, \eta), \ h^{-}, k^{-} \epsilon C^{1} \cap L_{1}; \]

\[ [(L_{\geq})^{*}]^{2} g^{-} = \]

\[ = -(L_{\geq})^{*} g^{-} [-b^{-} (\omega, \eta) (2ut)^{-1/2} + b_{3}^{-} (\omega, \eta)] - \]

\[ 2ut M_{-} (\omega, \eta) \nabla \ell_{-} \cdot \nabla (L_{\geq})^{*} g^{-} \]

\[ = g^{-} [-b^{-} (\omega, \eta) (2ut)^{-1/2} + b_{3}^{-} (\omega, \eta)]^{2} + 2ut M_{-} (\omega, \eta) \nabla \ell_{-} \cdot \]

\[ \nabla g^{-} [-b^{-} (\omega, \eta) (2ut)^{-1/2} + b_{3}^{-} (\omega, \eta)] \]

\[ + 2ut M_{-} (\omega, \eta) \nabla \ell_{-} \cdot \]

\[ \nabla \{ [-b^{-} (\omega, \eta) (2ut)^{-1/2} b_{3}^{-} (\omega, \eta)] g^{-} + 2ut M_{-} (\omega, \eta) \nabla \ell_{-} \cdot \nabla g^{-} \} \]

\[ = g^{-} (\eta; z) b^{-} (\omega, \eta)^{2} (2ut)^{-1} - 2ut g^{-} b^{-} (\omega, \eta) M_{-} (\omega, \eta) \nabla \ell_{-} \cdot \]

\[ \nabla (2ut)^{-1/2} + (2ut)^{1/2} \ell_{-} (\omega, \eta) + m_{-} (\omega, \eta) \]

\[ = g^{-} (\eta; z) b^{-} (\omega, \eta) [b^{-} (\omega, \eta) - M_{-} (\omega, \eta) \omega_{3} (t^{2} + u^{2})] (2ut)^{-1} \]

\[ -b^{-} (\omega, \eta) g^{-} (\eta; z) M_{-} (\omega, \eta) \omega' \eta' (2ut)^{-1/2} + (2ut)^{1/2} \ell_{-} (\omega, \eta) \]

\[ + m_{-} (\omega, \eta), \ \ell_{-}, m_{-} \epsilon C^{1} \cap L_{1}. \]
From (7.27), (7.28) we now observe that as \( u_0 \to 0 \) \( a(\omega, \eta) + \omega_3 M(\omega, \eta) (t^2 + u^2) \to -\omega_3^{-1} + \omega_3^{-1} = 0 \), \( b^- (\omega, \eta) - M_-(w, \eta) \omega_3 (t^2 + u^2) \to \omega_3^{-1} - \omega_3^{-1} = 0 \). We now write, e.g., in the integral over \([C^\leq_+(u_0)]\), the integrand times 2ut by \( h_(s, t, u) \), cancel the \( u \) in the denominator, and then return to Cartesian coordinates. Proceeding in this way, we may assert that

\[
k_{32}^2 (x; z) = 0 (|x|^{-2}).
\] (7.32)

As concerns \( k_{31}^s \) of (7.19), the first order of business is to examine the phases \( \eta_s^\pm \) for critical points on \( S^2 \). It is clear from the above that there can only be critical points for \( |q'| < n \). On \( S^2 \pm \) we consider coordinates \( q', \pm \sqrt{(1 - |q'|^2)} \) and write (cf. (7.23))

\[
\ell(\omega, \eta) = |\eta|F(\omega, q),
\]

\[
F(\omega, q) = \omega' q' \pm n \omega_3 \sqrt{(1 - n^{-2} |q'|^2)} = \omega \cdot (q', \pm n \sqrt{(1 - n^{-2} |q'|^2)}),
\]

recalling that \( F \) corresponds in the integrand to \( \mp q_3 > 0 \).

For \( |q'| < n \) we have

\[
\frac{\partial F}{\partial q_i} = \omega_i \mp n^{-1} \omega_3 q_i / \sqrt{(1 - n^{-2} |q'|^2)}, \quad i = 1, 2,
\]

critical points of \( F \) : \( q_3^\pm = (\pm n \omega', \mp \sqrt{(1 - n^2 |\omega'|^2)}) \),

\[
F(\omega, q_3^\pm) = \pm n |\omega'|^2 \pm n \omega_3 \sqrt{(1 - |\omega'|^2)} = \pm n,
\]

\[
\text{Hess } F(q_3^\pm) = \pm n^{-1} \omega_3^{-2} \begin{bmatrix} (1 - \omega_3^2) & \omega_1 \omega_2 \\ \omega_1 \omega_2 & (1 - \omega_1^2) \end{bmatrix}, \quad (7.33)
\]
\[ \sqrt{\det \text{Hess} F_\pm(q_s^\pm)} = n^{-1} \omega_3^{-1}, \]
\[ \exp[i \pi \text{sgn Hess } F_\pm(q_s^\pm)/4] = i, \]
\[ \theta^\pm(q_s^\pm) = t(\pm \omega), \]
\[ \Delta^\pm_s(q_s^\pm) = (1 - 2n^2|\omega'|^2) + 4n^2|\omega'|^2 \omega_3 \sqrt{(1 - n^2|\omega'|^2)}, \]
\[ \tilde{\Delta}_s^\pm(q_s^\pm) = (1 - 2n^2|\omega'|^2) - 4n^2|\omega'|^2 \omega_3 \sqrt{(1 - n^2|\omega'|^2)}, \]
\[ r_{ss}(q_s^\pm) = \tilde{\Delta}_s^\pm(q_s^\pm)/\Delta^\pm_s(q_s^\pm), \]
\[ v(q_s^\pm) = -|\omega'|^{-1} t(\omega' \sqrt{(1 - n^2|\omega'|^2)}, |\omega'|^2). \]

Now let \( \phi^\pm \in C_0^\infty(S^2) \) with \( \phi^\pm = 1 \) in neighborhoods of \( q_s^\pm \) and \( \text{supp} \phi^\pm \cap \{|q'| = n\} = \emptyset \) (\text{supp} \phi^\pm depends on \( |\omega'| < 1 \)). We set \( \tilde{\phi} = 1 - \phi^+ - \phi^- \); then in an obvious notation we have from (7.22)

\[ k_{31}^s(x; z) = k_{31}^s(x; z)^+ + k_{31}^s(x; z)^- + \tilde{k}_{31}^s(x; z) \tag{7.34} \]

It is straightforward to show that

\[ \tilde{k}_{31}^s(x; z) = 0_{\omega(|x|^{-2})}. \tag{7.35} \]

We proceed to the interesting cases: with \( j = \pm \)

\[ k_{31}^s(x; z)^j = 2^{-1}(2\pi)^{-3} \int K_{31}^j(x, y; z)f(y)dy, \]
\[
K_{31}^j(x, y; z) = \int_0^\infty (c_s^2 r^2 - z)^{-1} \psi(r) r^2 I_{31}^j(r, x, y) dr,
\]

(7.36)

\[
I_{31}^j(r, x, y) = \int_{S^2} \phi^j(q) \exp[ir|x| F_j(\omega, q) - iryq] \rho^j_p(q) \otimes v(q) dS.
\]

From (7.2), (7.33), (7.36)

\[
I_{31}^j(r, x, y) = i 2 \pi n \omega_3 r^j_p(q_s^j)(1 - n^2 |\omega'|^2)^{-1/2} \exp(i r t_j)(j \omega) \otimes v(q_s^j)
\]

\[
|rx|^{-1} + 0, (|rx|^{-2}),
\]

\[
t_j = j n|x| - q_s^j y,
\]

so that

\[
K_{31}^j(x, y; (\nu \pm i \epsilon)^2) = i j 2 \pi n \omega_3 r^j_p(q_s^j)(1 - n^2 |\omega'|^2)^{-1/2} \omega \otimes v(q_s^j)|x|^{-1}
\]

\[
I^\pm(t_j, \nu, \epsilon) + 0, (|x|^{-1-\kappa}), \kappa \in (1/2, 1), (7.37)
\]

where with \( \phi(r) = c_s^{-2} r \psi(r/c_s)(r + \nu \pm i \epsilon)^{-1} \)

\[
I^\pm(t_j, \nu, \epsilon) = \int_0^\infty [r - (\nu \pm i \epsilon)]^{-1} \phi(r) \exp(i r t_j/c_s) dr.
\]

In the usual way we now have

\[
I^+(t_j; \nu, \epsilon) = i(2\pi)^{1/2} \int_{-\infty}^{t_j/c_s} \exp[i(\nu + i \epsilon)(t_j/c_s - \tau)] \Phi_1^* \phi(\tau) d\tau,
\]

77
\[ I^-(t_j; \nu, \epsilon) = -i(2\pi)^{1/2} \int_{t_j/c_s}^{\infty} \exp[i(\nu - i\epsilon)(t_j/c_s - \tau)]\Phi_1^*(\tau) d\tau, \]

\[ I^\pm(t_{\pm 1}, \nu, 0) = \pm i2\pi c_s^{-2}\nu(2\nu)^{-1} \exp[i\nu(\pm n|x| - q_s^\pm y)/c_s] + 0(|x|^{-1}), \]

\[ I^\pm(t_{\mp 1}, \nu, 0) = 0(|x|^{-1}). \]

Hence,

\[ K^{\mp}_{31}(x, y; (\nu \pm i0)^2) = -(2\pi)^2 c_s^{-2} 2^{-1} n \omega_3 r_{ps}^\pm (q_s^\pm)(1 - n^2|\omega'|^2) \]

\[ \text{exp}[i\nu(\pm n|x| - q_s^\pm y)/c_s] |x|^{-1} \omega \otimes v(q_s^\pm) + 0_\omega(|x|^{-1-\kappa}), \quad (7.38) \]

\[ K^{\mp}_{31}(x, y; (\nu \pm i0)^2) = 0_\omega(|x|^{-1-\kappa}), \quad \kappa \in (1/2, 1). \]

Thus, from (7.36), (7.38)

\[ k_{31}(x; (\nu \pm i0)^2)^\pm = \]

\[ = -4^{-1}(2\pi)^{-1} c_s^{-2} n \omega_3 r_{ps}^\pm (q_s^\pm)(1 - n^2|\omega'|^2)^{-1/2} \]

\[ \text{exp}[(\pm i\nu|x|/c_p)|x|^{-1} \omega \otimes v(q_s^\pm) \int \exp[-i\nu q_s^\pm y/c_s] \times \]

\[ f(y) dy \quad (7.39) \]

\[ = -g_p^\pm(x; \nu)r_{ps}^\pm (q_s^\pm) \omega \otimes v(q_s^\pm) K_3^s(\omega, \nu) + 0_\omega(|x|^{-1-\kappa}), \]

78
\[ K_3^s(\omega, \nu) = 2^{-1}(c_p c_s)^{-1}(1 - n^2|\omega'|^2) \]

\[ \int \exp(-i\nu q_s y/c_s) f(y) dy, \]

\[ g_p^\pm(x; \nu) = (4\pi|x|)^{-1} \exp(\pm i\nu |x|/c_p), \kappa \in (1/2, 1), \]

\[ k_{31}^s(x; (\nu \mp i0)^2)^\pm = 0_\omega(|x|^{-1-\kappa}). \]

We thus have

\[ k_{31}(x; (\nu \pm i0)^2) = \]

\[ = -g_p^\pm(x; \nu) r_{ps}(q_s^\pm) \omega \otimes v(q_s^\pm) K_3^s(\omega; \nu) + 0_\omega(|x|^{-1-\kappa}). \quad (7.40) \]

From (7.32), (7.40)

\[ k_3(x; (\nu \pm i0)^2) = \]

\[ = g_p^\pm(x; \nu) r_{ps}(q_s^\pm) \omega \otimes v(q_s^\pm) K_3^s(\omega; \nu) + 0_\omega(|x|^{-1-\kappa}). \quad (7.41) \]

Finally, from (7.12), (7.14), (7.21), (7.41)

\[ k^s(x; (\nu \pm i0)^2) = \]

\[ = g_s^\pm(x; \nu) v(\omega) \otimes v(\omega) K_1^s(\pm \omega; \nu) - \]

\[ g_s^\pm(x; \nu) r_{ss}(q_s^\pm) \omega \otimes v(\tilde{\omega}) K_2^s(\pm \tilde{\omega}; \nu) + \]

\[ g_p^\pm(x; \nu) r_{ps}(q_s^\pm) \omega \otimes v(q_s^\pm) K_3^s(\omega; \nu) + 0_\omega(|x|^{-1-\kappa}), \quad (7.42) \]

\[ \kappa \in (1/2, 1). \]

In the same manner from (7.12) we have
\begin{align*}
\ell^s(x; (\nu \pm i0)^2) &= \\
&= g^\pm_s(x; \nu) \nu^\pm_{ss}(\nu) v(\nu) \otimes v(\nu^\pm) L^s_1(\nu; \nu) - \\
&\quad g^\pm_s(x; \nu) \nu^\pm_{ss}(\nu) v(\nu) \otimes v(\nu) L^s_2(\nu; \nu) + \\
&\quad g^\pm_p(x; \nu) \nu^\pm_{ps}(q^\pm_{ss}) \nu^\pm_{ss}(q^\pm_s) \omega \otimes v(\nu^\pm) L^s_3(\omega; \nu)
\end{align*}

\begin{align*}
L^s_1(\omega; \nu) = L^s_2(\omega; \nu) = 2^{-1} c_s^{-2} \int \exp(i\nu \omega y / c_s) f(y) dy, \\
L^s_3(\omega; \nu) = 2^{-1} (c_p c_s)^{-1} \omega_3 (1 - n^2 |\omega'|^2) \int \exp(i\nu q^\pm_{ss} y / c_s) f(y) dy.
\end{align*}

Also,

\begin{align*}
m(x; (\nu \pm i0)^2) &= \\
&= g^\pm_s(x; \nu) \nu^\pm_{ps}(\nu) v(\nu) \otimes \theta^\pm(\nu) M^s_1(\nu; \nu) - \\
&\quad g^\pm_s(x; \nu) \nu^\pm_{ps}(\nu) \nu^\pm_{ss}(\nu) v(\nu) \otimes \theta^\pm(\nu) M^s_2(\nu; \nu) + \\
&\quad g^\pm_p(x; \nu) \nu^\pm_{ps}(q^\pm_{ss}) \nu^\pm_{ss}(q^\pm_s) \omega \otimes M^s_3(\omega; \nu) + 0_\omega(|x|^{-1-\kappa}).
\end{align*}

\begin{align*}
M^s_1(\omega; \nu) &= 2^{-1} c_s^{-2} \int \exp[-i\nu y (\omega', n \theta^\pm_3(\omega') / c_s)] f(y) dy, \\
M^s_2(\omega; \nu) &= 2^{-1} c_s^{-2} \int \exp[-i\nu y (\omega', n \theta^\pm_3(\omega') / c_s)] f(y) dy, \\
M^s_3(\omega; \nu) &= 2^{-1} (c_p c_s)^{-1} \omega_3 (1 - n^2 |\omega'|^2)^{-1/2} \times
\end{align*}
Finally, from (7.10), (7.12), (7.42), (7.43), (7.44)

\begin{align*}
v_{sv}(x; (\nu \pm i0)^2) &= g_s^\pm(x; \nu) v(\nu) F^\pm_{sv}(\omega; \nu) + g_p^\pm(x; \nu) \iota \omega F^\pm_{ps}(\omega; \nu) \\
&+ 0(|x|^{-1-\kappa}),
\end{align*}

(7.45)

where the values of \( F^\pm_{sv} \) and \( F^\pm_{ps} \) can be read off from the aforementioned formulas. Observe that we have written \( 0(|x|^{-1-\kappa}) \) in place of \( 0_\omega(|x|^{-1-\kappa}) \), since the latter is equal to a sum of bounded functions of \( \omega \). The original apparent anistropy of the estimate occurred only because of our crude methods of analysis. In the end it disappears.

8 The Steady-State SVP Wave

The component of the solution \( v_p(x; (\nu \pm i0)^2) \) consists of a \( P \) wave due to the incident and reflected \( P \) modes and an \( SV \) wave due to \( S \) modes created at the boundary \( \{x_3 = 0\} \). We call this the \( SVP \) wave. In this section we establish the existence and asymptotics of \( SVP \).

Let \( \nu \in (\nu_0 - \delta, \nu_0 + \delta), \nu_0 - 4\delta > 0, \psi \in C_0^\infty(\mathbb{R}), \supp \psi \subset \{ r : |c_p r - \nu_0| \leq 4\delta \} \subset (0, \infty), \psi(r) = 1 \) for \( |c_p r - \nu_0| \leq 3\delta, \chi \in C_0^\infty(\mathbb{R}^3 \setminus \{0\}), \chi(\eta) = \chi(r q) \equiv \psi(r), q \in S^2, z = (\nu \pm i\eta)^2, \epsilon \in (0, \epsilon_0) \). In the notation of (4.3) let

\begin{align*}
\pi^\pm_p(x, \nu) &= \exp(i\nu \eta^t q + \exp(i \eta \tilde{q}) r_{pp}(q) \eta^t \tilde{q} - \exp(i \nu \eta^t \eta) r_{sp}(q) \eta^t \phi). \\
\end{align*}

(8.1)
Then from (4.3), (4.11), (5.3)

\[ v_p(x; z) = 2^{-1}[(\Psi_p^+)^* (c_p^2 \cdot |^2 - z)^{-1}\Psi_p^+ f(x) + (\Psi_p^-)^* (c_p^2 \cdot |^2 - z)^{-1}\Psi_p^- f(x)] \] (8.2)

\[ \equiv v_p^1(x; z) + v_p^2(x; z), \]

\[ v_p^1(x; z) \equiv v_p^1(x; z; \psi) = 2^{-1}(2\pi)^{-3/2} \int \int (c_p^2 r^2 - z)^{-1}\psi(r) \{ \chi_-(q_3) \]

\[ \pi_p^+ (x; \eta) \otimes [q \Phi f(\eta) - r_{pp}(q)\bar{q}\Phi f(\bar{\eta}) - r_{sp}(q)v^+(\phi)\Phi f(\eta_p^+)] \]

\[ + \chi_-(q_3)\pi_p^- (x; \eta) \otimes [q \Phi f(\eta) - r_{pp}(q)\bar{q}\Phi f(\bar{\eta}) - r_{sp}(q)v^- (\phi)\Phi f(\eta_p^-)] \} r^2 dS dr, \]

\[ v_p^2(x; z) \equiv v_p^1(x; z; 1 - \psi). \]

Replacing the interval \(|c_p r - \nu_0| \leq 2\delta\) by a circle of radius \(2\delta < \epsilon_0\) in the lower \((z = (\nu + i\epsilon)^2)\) or upper \((z = (\nu - i\epsilon)^2)\) half plane and extending \(\psi\) to this half disk by one, it follows that \(v_p^1(x; (\nu \pm i0)^2)\) exists and is continuous in \((x, \nu)\), \(\nu \in [\nu_0 - \delta, \nu_0 + \delta]\). Hence \(v_p^1(x; (\nu \pm i\epsilon)^2)\) exists and is bounded and continuous on \(\mathbb{R}_+^2 \times [\nu_0 - \delta, \nu_0 + \delta] \times [0, \epsilon_0].\) Since the integrand of \(v_p^2(x; z)\) contains no singularity at \(c_p r = \nu \pm i0,\) the same applies to it. In summary, \(v_p(x; z)\) exist and is bounded and continuous on \(\mathbb{R}_+^2 \times [\nu_0 - \delta, \nu_0 + \delta] \times [0, \epsilon_0].\) The same obviously applies to \(D^3 v_p(x; z),\) i.e., the principle
of limiting absorption holds.

To study the asymptotics of \( v_p(x; (\nu \pm i0)^2) \) we write (8.2) as (cf (7.9))

\[
v_p(x; z) = k^p(x; z) - \ell^p(x; z) - m^F(x; z),
\]

\[
k^p(x; z) =
\]

\[= 2^{-1}(2\pi)^{-3/2} \int (c^2_p|\eta|^2 - z)^{-1} [\chi_-(q_3) \pi^+_{p^*}(x, \eta) + \chi_+(q_3) \pi^-_{p^*}(x, \eta)] \otimes q\Phi f(\eta)d\eta
\]

\[= 2^{-1}(2\pi)^{-3/2} \int (c^2_p|\eta|^2 - z)^{-1} \{\exp(ix\eta)q - \exp(i\bar{x}\eta)r_{pp}(q)\bar{q} - r^\pm_{sp}(q)\chi_-(q_3)\exp(ix\eta_p^+)v^+(\phi) + \chi_+(q_3)\exp(ix\eta^-_p)v^-(\phi)\} \otimes q\Phi f(\eta)d\eta
\]

\[= k^p_1(x; z) + k^p_2(x; z) + k^p_3(x; z),
\]

\[
\ell^p(x; z) =
\]

\[= 2^{-1}(2\pi)^{-3/2} \int (c^2_p|\eta|^2 - z)^{-1} [\chi_-(q_3) \pi^+(x, \eta) + \chi_+(q_3) \pi^-(x, \eta)]
\]

\[= 2^{-1}(2\pi)^{-3/2} \int (c^2_p|\eta|^2 - z)^{-1} \{\exp(ix\eta)q - \exp(i\bar{x}\eta)r_{pp}(q)\bar{q} - r^\pm_{sp}(q)\chi_-(q_3)\exp(ix\eta_p^+)v^+(\phi) + \chi_+(q_3)\exp(ix\eta^-_p)v^-(\phi)\} \otimes
\]

83
\[ r_{pp}(q) \tilde{q} \Phi f(\tilde{\eta}) d\eta \]

\[ \equiv \ell_{1}^{p}(x; z) + \ell_{2}^{p}(x; z) + \ell_{3}^{p}(x; z), \]

\[ m^{p}(x; z) = -2^{-1}(2\pi)^{-3/2} \int (\epsilon^{2}_{p}|\eta|^{2} - z)^{-1} \{ \chi_{-}(q_{3}) \pi^{+}(x, \eta) \otimes v^{+}(\phi) \Phi f(\eta_{p}^{+}) + \chi_{+}(q_{3}) \pi^{-}(x, \eta) \otimes v^{-}(\phi) \Phi f(\eta_{p}^{-}) \} r_{sp}(q) d\eta \]

\[ = 2^{-1}(2\pi)^{-3/2} \int (\epsilon^{2}_{p}|\eta|^{2} - z)^{-1} \{ \exp(ix\eta)r_{sp}^{\pm}(q) \otimes [\chi_{-}(q_{3}) \pi^{+}(x, \eta) \otimes v^{+}(\phi) \Phi f(\eta_{p}^{+}) + \chi_{+}(q_{3}) \pi^{-}(x, \eta) \otimes v^{-}(\phi) \Phi f(\eta_{p}^{-}) \} \}

\[ \equiv m_{1}^{p}(x; z) + m_{2}^{p}(x; z) + m_{3}^{p}(x; \eta). \]

As in §7, we treat \( k^{p} \) in all detail; the computations for \( \ell^{p} \) and \( m^{p} \) then follow one of the patterns established. From (8.3) with the functions \( \psi \) and \( \chi \) introduced above we have

\[ k_{1}^{p}(x; z) = k_{11}^{p}(x; z) + k_{12}^{p}(x; z), \]

\[ k_{12}^{p}(x; z) = 2^{-1}(2\pi)^{-3/2} \int (\epsilon^{2}_{p}|\eta|^{2} - z)^{-1} [1 - \chi(\eta)] \]

\[ r_{sp}(q) \tilde{q} \Phi f(\tilde{\eta}) d\eta \]

\[ \Rightarrow \ell_{1}^{p}(x; z) + \ell_{2}^{p}(x; z) + \ell_{3}^{p}(x; z), \]

\[ m^{p}(x; z) = -2^{-1}(2\pi)^{-3/2} \int (\epsilon^{2}_{p}|\eta|^{2} - z)^{-1} \{ \chi_{-}(q_{3}) \pi^{+}(x, \eta) \otimes v^{+}(\phi) \Phi f(\eta_{p}^{+}) + \chi_{+}(q_{3}) \pi^{-}(x, \eta) \otimes v^{-}(\phi) \Phi f(\eta_{p}^{-}) \} r_{sp}(q) d\eta \]

\[ = 2^{-1}(2\pi)^{-3/2} \int (\epsilon^{2}_{p}|\eta|^{2} - z)^{-1} \{ \exp(ix\eta)r_{sp}^{\pm}(q) \otimes [\chi_{-}(q_{3}) \pi^{+}(x, \eta) \otimes v^{+}(\phi) \Phi f(\eta_{p}^{+}) + \chi_{+}(q_{3}) \pi^{-}(x, \eta) \otimes v^{-}(\phi) \Phi f(\eta_{p}^{-}) \} \}

\[ \equiv m_{1}^{p}(x; z) + m_{2}^{p}(x; z) + m_{3}^{p}(x; \eta). \]
\[
\exp(ix\eta)q \otimes q\Phi f(\eta)d\eta \\
= 0(|x|^{-3+\nu}), \mu > 0, \quad (8.4)
\]

\[
k^p_{11}(x; z) = 2^{-1}(2\pi)^{-3/2} \int K_{11}(x; y; z)f(y)dy,
\]

\[
K_{11}(x; y; z) = \int_0^\infty \int_{S^2} (c^2r^2 - z)^{-1}\psi(r) r^2 \exp[ir(x - y)]q \otimes q dSdr,
\]

so that exactly as in (7.14)

\[
k^p_1(x; (\nu \pm i0)^2) = g^\pm_p(x; \nu)\omega \otimes \omega K^p_1(\pm\omega; \nu) + 0(|x|^{-1-\kappa}), \quad (8.5)
\]

\[
K^p_1(\omega; \nu) = 2^{-1}c^{-2}_p \int \exp(-i\nu\omega y/c_p)f(y)dy, \kappa \in (1/2, 1).
\]

In the same way

\[
k^p_2(x; (\nu \pm i0)^2) = -g^\pm_p(x; \nu)r_{pp}(\omega)\omega \otimes \bar{\omega}K^p_2(\pm\bar{\omega}; \nu) + 0(|x|^{-1-\kappa}),
\]

\[
K^p_2 = K^p_1, \quad \kappa \in (1/2, 1). \quad (8.6)
\]

Finally,

\[
-k^p_3(x; z) = k^p_{31}(x; z) + k^p_{32}(x; z),
\]

\[
k^p_{31}(x; z) \equiv k^p_{31}(x; z; \psi) = 2^{-1}(2\pi)^{-3} \int K_{31}(x; y; z)f(y)dy,
\]

\[
K_{31}(x; y; z) = \int_0^\infty (c^2r^2 - z)^{-1}\psi(r) r^2 dr \int_{S^2} \{r^+_sp(q) \times
\]

85
\[ [\chi_-(q_3) \exp(irxn^{-1}\phi^+ - iryq)v^+ (\phi)] + r_{sp}^{-}(q)[\chi_+(q_3) \exp(irxn^{-1}\phi^- - iryq)v^- (\phi)] \} \otimes qdS_q, \]

\[ k_{32}^p (x; z) = k_{31}^p (x; z; 1 - \psi). \]

To find the critical points of the phase in \( K_{31} \), we write

\[ G_\pm (\omega, q) = n^{-1} \omega \phi^\pm = \omega' q' \pm \omega_3 \sqrt{(1 - n^2|q'|^2)}, \]

(8.8)

where \( G_\pm \) corresponds to \( \mp \sqrt{(1 - |q'|^2)} \). Comparing (7.33) \( (F_\pm \text{ with } n \rightarrow n^{-1} \text{ is } G_\pm) \), we thus have critical points only for \( |\omega'| < n \), and these are (see also (4.5))

\[ \frac{\partial G_\pm}{\partial q_i} = \omega_i \mp n \omega_3 q_i \sqrt{(1 - n^2|q'|^2)}, \]

\[ q^\pm_\omega = (\pm n^{-1} \omega', \mp \sqrt{(1 - n^{-2}|\omega'|^2)}), \]

\[ G_\pm (\omega, q^\pm_\omega) = \pm n^{-1}, \]

\[ \sqrt{|\det \text{ Hess } G_\pm (\omega, q^\pm_\omega)|} = n \omega_3^{-1}, \]

(8.9)

\[ \exp(i\pi \text{ sgn Hess } G_\pm (\omega, q^\pm_\omega)/4) = i, \]

\[ v^\pm(\phi(q^\pm_\omega)) = v(\omega), \]

\[ \Delta_p(q^\pm_\omega) = (1 - 2|\omega'|^2) + 4n|\omega'| \sqrt{(1 - |\omega'|^2)} \sqrt{(1 - n^{-2}|\omega'|^2)}, \]

\[ \tilde{\Delta}_p(q^\pm_\omega) = (1 - 2|\omega'|^2) - 4n|\omega'| \sqrt{(1 - |\omega'|^2)} \sqrt{(1 - n^{-2}|\omega'|^2)}, \]

\[ r_{sp}^\pm (q^\pm_\omega) = \mp 4n^{-1}|\omega'| (1 - 2|\omega'|^2) \sqrt{(1 - n^{-2}|\omega'|^2)} / \Delta_p(q^\pm_\omega) \]

86
\[ r_{pp}(q^\pm_p) = \tilde{\Delta}_p(q^\pm_p)/\Delta_p(q^\pm_p), \]
\[ |n(q^\pm_p)| = (1 - n^{-2}|\omega'|^2)^{-1/2}. \]

Let \( \lambda^\pm \in C_0^\infty(S^2) \), \( \lambda^\pm = 1 \) in a neighborhood of \( q^\pm_p \), \( \text{supp} \lambda^\pm \cap \{q_3 = 0\} = \emptyset \), \( \tilde{\lambda} = 1 - \lambda^+ - \lambda^- \) (\( \text{supp} \lambda^\pm \) depends on \( |\omega'| < n \)). Then from (7.2), (8.7) for \( |\omega'| < n \)

\[ K_j^{31}(x, y; z) = K_{31}^+(x, y; z) + K_{31}^-(x, y; z) + \tilde{K}_{31}(x, y; z), \]
\[ K_{31}^j(x, y; z) = \int (c_p^2 r^2 - z)^{-1} \psi(r) r^2 I^j(x, y, r) dr, \]
\[ I^j(x, y; z) = \int_{S^2} \chi^j(q) r_{sp}^j(q) \exp[ir|x|G_j(\omega, q) - irqy]v^j(q) \otimes qdS_q \]
\[ = i2\pi\omega^j_n^{-1}r_{sp}^j(q_p^\pm)(1 - n^{-2}|\omega'|^2)^{-1/2} \exp(irt_j)v(\omega) \otimes q_p^j \]
\[ |r|x|^j - 1 + O(|r|x|^{-2}), \quad t_j = jn^{-1}|x| - yq^j_p. \]

From this we have

\[ K_{31}^j(x, y; (\nu \pm i\epsilon))^2) = i2\pi n^{-1}\omega^3 r_{sp}^j(q_p^\pm)(1 - n^{-2}|\omega'|^2)v(\omega) \otimes q_p^j|x|^{-1} \]
\[ \int_0^\infty [r - (\nu \pm i\epsilon)]^{-1}\phi(r) \exp(irt_j/c_p) dr \]
\[ \phi(r) = c_p^{-2}\psi(r/c_p)(r + \nu \pm i\epsilon)^{-1}, \]

87
so that \((j = \pm)\)

\[
K_{31}^j(x, y; \nu + j i 0) = - j (2\pi)^2 (c_p c_s)^{-1} \omega_3 r_{sp}^j(q_p^i)(1 - n^{-2} |\omega'|^2)^{-1/2}
\]

\[
v(\omega) \otimes q_p^i |x|^{-1} \exp[i \nu (jn - 1 |x| - q_p^j y)/c_p] +
\]

\[
0_\omega(|x|^{-1-\kappa}), \tag{8.11}
\]

uniformly with respect to \(y\) in compact sets.

To estimate \(\tilde{K}_{31}(x, y; z)\) it suffices to estimate an integral of the form

\[
I(r, x, y) = \int_{\{q_3 < 0\}} \lambda(q) r_{sp}(q) \exp(ix\eta_p^+ - irqy) v^+(\phi) \otimes q dS_q +
\]

\[
\int_{\{q_3 > 0\}} \lambda(q) r_{sp}(q) \exp(ix\eta_p^- - irqy) v^-(\phi) \otimes q dS_q,
\]

where \(\lambda\) is smooth with support in a neighborhood of \(\{q_3 = 0\}\) which does not intersect the supports of \(\lambda^\pm\); it is assumed to be equal to one in a smaller neighborhood. Let

\[
h^\pm(q) \equiv h^\pm(r, y, q) = [\lambda(q) r_{sp}(q) \exp(-irqy) v^\pm(\phi) \otimes q]_{ij},
\]

\[
i, j = 1, 2, 3,
\]

\[
h^\pm(q', 0^+) = 0 \text{ (see (3.5))}.
\]

In polar coordinates \(0 < \theta < \pi, 0 < \phi < 2\pi\) with \(G^\pm(\omega, q)\) of (8.8)

\[
\exp(ix\eta_p^\pm) = \exp(ir|x|G^\pm(\omega, q)),
\]
\[ G_{\pm}(\omega, q) = \omega_1 \sin \theta \cos \phi + \omega_2 \sin \theta \sin \phi \pm \omega_3 n^{-1} \sqrt{1 - n^2 \sin \theta}, \]

\[ \nabla_q G_{\pm}(\omega, q) \neq 0 \text{ on supp } \lambda \setminus \{ q_3 = 0 \}, \]

\[ \lim \nabla_q G_{\pm}(\omega, q) \neq 0 \text{ as } |q'| \to 1 \text{ for } |\omega'| < n, \]

\[ L_{\pm} = |\nabla_q G_{\pm}|^{-2} \nabla_q G_{\pm} : \nabla_q = \]

\[ = |\nabla_q G_{\pm}|^{-2}[(\partial_\theta G_{\pm}) \partial_\theta + (\sin \theta)^{-2}(\partial_\phi G_{\pm}) \partial_\phi], \]

\[ L_{\pm}^* h_{\pm} = \]

\[ -(1/ \sin \theta) \{ \partial_\theta[\sin \theta h_{\pm}|\nabla_q G_{\pm}|^{-2}(\partial_\theta G_{\pm})] - \partial_\phi[(\partial_\phi G_{\pm})|\nabla_q G_{\pm}|^{-2}h_{\pm}] \}, \]

\[ dS = \sin \theta d\theta d\phi. \]

We thus have

\[ ir|x|I(x, y, r) = \]

\[ = \int_{S^2} h_+(q) L_+ \exp[ir|x|G_+(\omega, q)]dS + \int_{S^2} h_-(q) L_- \exp[ir|x|G_-(\omega, q)]dS \]

\[ = \int_{S^2} \exp[ir|x|G_+(\omega, q)] L_+^* h_+(q)]dS + \int_{S^2} \exp[ir|x|G_-(\omega, q)] L_-^* h_-(q)dS \]

\[ (8.12) \]
\[-r^2|x|^2I(x, y, r) =
\]
\[
= \int_{S^2_+} L_+^* h_+(q) \cdot L_+ \exp[ir|x|G_+(\omega, q)]dS +
\]
\[
\int_{S^2_-} L_-^* h_-(q) \cdot L_- \exp[ir|x|G_-(\omega, q)]dS 
\]
\[
= \int |\nabla G_+|^2 [L_+^* h_+(q)](\partial_\theta G_+) \exp(ir|x|G_+)|\theta=\pi/2d\phi +
\]
\[
\int |\nabla G_-|^2 [L_-^* h_-(q)](\partial_\theta G_-) \exp(ir|x|G_-)|\theta=\pi/2d\phi +
\]
\[
\left\{ \int \begin{array}{ll}
{\{q_3 < 0} & \exp[ir|x|G_+(\omega, q)](L_+^*)^2h_+(q)dS + \\
{\{q_3 > 0} & \exp[ir|x|G_-(\omega, q)](L_-^*)^2h_-(q)dS 
\end{array} \right\}.
\]

Now \(L_+^* h_+(q)\) is bounded at \(\theta = \pi/2\), and \(\partial_\theta G_+(\omega, q) = 0\) there. The term in braces is bounded by a constant \(c = c(\omega) < \infty, |\omega'| < n\), uniformly with respect to \(r, y\) in compact sets. Thus from (8.7), (8.10), (8.11), (8.12) for \(|\omega'| < n\) uniformly with respect to \(y\) in compact sets

\[
\tilde{K}_{31}(x, y; (\nu \pm i0)^2) = 0, (|x|^{-1-\kappa}),
\]

\[
K_{31}(x, y; (\nu \pm i0)^2) = \mp(2\pi)^2(c_p c_s)^{-1}a_{3p} r_{sp}(q^\pm_{p})(v(\omega) \otimes q^\pm_{p})|x|^{-1}
\]

90
\[ \exp[i\nu(|x| - q^\pm_p y)/c_p] + 0_\omega(|x|^{-2-\kappa}). \]

For \(|\omega'| > n\) the phases in the integrals of (8.7) have no critical points, and \(K_{31}(x, y; z)\) can be estimated just as \(\bar{K}_{31}\) above. Thus, finally from (8.7), (8.14)

\[ k_{31}^p(x; (\nu \pm i0)^2) = \mp \chi_{(0,n)}(|\omega'|)g_\pm_s(x; \nu)r_{sp}(q^\pm_p)\nu(\omega) \otimes q^\pm_p K_3^p(\omega; \nu) \]

\[ + 0_\omega(|x|^{-1-\kappa}), \quad (8.15) \]

\[ K_3^p(\omega, \nu) = 2^{-1}(c_pc_s)^{-1}\omega_3(1-n^{-2}|\omega'|^2)^{-1/2} \int \exp(-i\nu q^\pm_p y/c_p)f(y)dy. \]

In the now familiar way

\[ k_{32}^p(x; z) = 0_\omega(|x|^{-2}), \omega_3 \neq n, \quad (8.16) \]

uniformly with respect to \(\nu \in [\nu_0 - \delta, \nu_0 + \delta], \epsilon \in (0, \epsilon_0]\), where the condition \(\omega_3 \neq n\) enters due to our crude methods; it goes away in the end.

From (8.7), (8.15), (8.16) we have for \(\omega_3 \neq n\)

\[ k_{33}^p(x; (\nu \pm i0)^2) = \mp \chi_{(0,n)}(|\omega'|)g^\pm_s(x; \nu)r_{sp}(q^\pm_p)\nu(\omega) \otimes q^\pm_p K_3^p(\omega; \nu) \]

\[ + 0_\omega(|x|^{-1-\kappa}). \quad (8.17) \]

Thus, from (8.3), (8.5), (8.6), (8.17)

\[ k_3^p(x; (\nu \pm i0)^2) = g^\pm_p(x; \nu)\omega \otimes [\omega K_1^p(\pm\omega; \nu) - r_{pp}(\omega) \bar{\omega} K_2^p(\pm\omega; \nu)] \]

\[ \pm \chi_{(0,n)}(|\omega'|)g^\pm_s(x; \nu)r_{sp}(q^\pm_p)\nu(\omega) \otimes q^\pm_p K_3^p(\omega; \nu) + 0_\omega(|x|^{-1-\kappa}), \quad \kappa \in (1/2, 1), \omega \neq n. \quad (8.18) \]
We note that from (8.9) $r_{sp} \to 0$ as $|\omega'| \to n$, so that the asymptotics are continuous in this sense.

In a similar manner from (8.3)
\[
\ell^p(x; (\nu \pm i0)^2) = \\
= g^\pm_p(x; \nu) \omega \otimes [\tilde{\omega} r_{sp}(\omega) L^p_1(\pm \omega; \nu) - \omega r^2_{pp}(\omega) L^p_2(\pm \omega; \nu)] \pm \\
\chi(0, n)(|\omega'|) g^\pm_s(x; \nu) v(\omega) \otimes [\tilde{q}^\pm_p r_{sp}(\tilde{q}^\pm_p) r_{pp}(\tilde{q}^\pm_p)] (8.19) \\
L^p_1 = L^p_2 = K^p_1, \quad L^p_3 = K^p_2,
\]
and also from (8.3)
\[
m^p(x; (\nu \pm i0)^2) = \\
= g^\pm_p(x; \nu) \omega \otimes [r^\pm_{sp}(\omega) v(\phi(\omega)) M^p_1(\omega; \nu) \mp r^\pm_{sp}(\omega) r_{pp}(\omega) v(\phi(\omega)) M^p_2(\omega; \nu)] \pm \\
\chi(0, n)(|\omega'|) g^\pm_s(x; \nu) r^2_{sp}(\tilde{q}^\pm_p) v(\omega) \otimes v(\omega) M^p_3(\omega; \nu) + \\
0_{\nu}(|x|^{-1-\kappa}), (8.20)
\]
\[
M^p_1(\omega; \nu) = 2^{-1} c_p^{-2} \int \exp[i\nu(\mp \omega', \pm n^{-1} \phi_3(\omega'))/c_p] f(y) dy, \\
M^p_2(\omega; \nu) = 2^{-1} c_p^{-2} \int \exp[i\nu(\mp \omega', \mp n^{-1} \phi_3(\omega'))/c_p] f(y) dy, \\
M^p_3(\omega; \nu) = 2^{-1}(c_p c_s)^{-1} \omega_3(1 - n^{-2} |\omega'|^2)^{-1/2}
\]
\[
\int \exp(\mp iv\omega y/c_s) f(y) dy. \tag{8.21}
\]

Note that in the third term above \(r_{sp}\) contains the factor 
\((1 - n^{-2}|\omega'|^2)^{1/2}\) which cancels the factor \((1 - n^{-2}|\omega'|^2)^{-1/2}\), so that there is actually no singularity.

From (8.18)-(8.20)
\[
v_p(x; (\nu \pm i0)^2) = g_{p}^{\pm}(x; \nu) t_{\omega} F_{p}^{\pm}(\omega; \nu) + g_{s}^{\pm}(x; \nu) v(\omega) \chi(0, n)(|\omega'|) F_{sp}^{\pm}(\omega; \nu) + O(|x|^{-1-\kappa}), \tag{8.22}
\]
where \(\kappa \in (1/2, 1)\), the values of \(F_{p}^{\pm}\) and \(F_{sp}^{\pm}\) can be read off from (8.18)-(8.20), and we point out that we have written 0 in place of \(0_{\omega}\), since the latter is equal to a sum of bounded functions of \(\omega\) defined for all \(\omega \in S^2_+\).

\section{The Uniqueness Theorem}

The purpose of the present section is to formulate a uniqueness class for the solutions \(v_{\pm}(x; \nu) = v(x; \nu \pm i0)^2\) of 
\([M(D) - \nu^2 I]v_{\pm}(x; \nu) = f(x) \in D(\mathbb{R}^3_+), B(D)v_{\pm}(x', 0; \nu) = 0\) constructed in \S\S 5-8. To do so, we first recall the properties of the smooth, bounded \(v_{\pm}(x; \nu)\) constructed and their properties and asymptotics. We present this summary as

\textbf{Theorem 9.1} From (5.3), (6.12), (7.7), (7.45), (8.22) with 
\(F_{j}^{\pm} = F_{j}^{\pm}(\omega; \nu; f)\) a bounded function of \((\omega, \nu), j \in \{sh, sv, sp, p, ps, R\}, \kappa \in \mathbb{R}_+\)
(1/2, 1), \kappa' \in (0, 1/2), a > 0

v_{\pm}(x; \nu) = v^\pm_s(x; \nu) + v^\pm_p(x; \nu) + v^\pm_R(x; \nu),

v^\pm_s(x; \nu) + v^\pm_p(x; \nu) = \tilde{v}^\pm_s(x; \nu) + \tilde{v}^\pm_p(x; \nu) + O(|x|^{-1-\kappa}),

\tilde{v}^\pm_s(x; \nu) = g^\pm_s(x; \nu)\{h(\omega)F_{sh}^\pm + v(\omega)[F_{sv}^\pm + \chi_{(0,n)}(|\omega'|)F_{sp}^\pm]\},

\tilde{v}^\pm_p(x; \nu) = g^\pm_p(x; \nu)^t\omega(F_p^\pm + F_{ps}^\pm), \quad (9.1)

v^\pm_R(x; \nu) = \gamma^\pm(x)|x|^{-1/2}F_{R}^\pm + R^\pm(x; \nu),

g^\pm_{p,s}(x; \nu) = (4\pi|x|)^{-1}\exp(\pm i\nu|x|/c_{p,s}),

|R^\pm(x; \nu)| \leq \text{const} \left(1 + x_3\right)\exp(-ax_3)|x'|^{-1-\kappa}'.

These components have the property that

\[ [M(D) - \nu^2 I][v^\pm_p(x; \nu) + v^\pm_s(x; \nu)] = \Psi_p\Psi_p f + \Psi_s\Psi_s f \]

is orthogonal in \(H\) to \([M(D) - \nu^2 I]v^\pm_R(x; \nu) = \Sigma^*\Sigma f\). Furthermore,

\[ tA(\omega)E_0^{-1}A(D)^\circ_{p,s}(x; \nu) = \pm i\nu c_{p,s}v^\circ_{p,s}(x; \nu) + O(|x|^{-2}), \]

\[ \omega = x/|x|, \]

\[ \pm \int_0^\infty i\gamma_\pm(x)A(\alpha, 0)E_0^{-1}A(D)\gamma_\pm(x)dx_3 = \text{const} > 0, \]

independent of \(x', \alpha = x'/|x'|\). Also, \(B(D)v^\pm_{p,s,R}(x', 0; \nu) = 0\).
Our task is to define classes $\mathcal{R}_\nu^\pm$ of functions with the properties listed above and to show that any solution of 

$$[M(D) - \nu^2 I]w_\pm = 0, \ w_\pm \in \mathcal{R}_\nu^\pm,$$

is zero. In so doing, we shall not strive for maximum generality but rather simply get the job done.

**Definition 9.1** For $\nu > 0$ $w_\pm \in \mathcal{R}_\nu^\pm$ iff

1. $w_\pm \in C^2(\mathbb{R}_+^3, \mathbb{C}^3)$, $w_\pm = w_\pm^s + w_\pm^p + w_\pm^R$, 
   $[M(D) - \nu^2 I](w_\pm^s + w_\pm^R) \in \mathcal{H}$ is orthogonal to $[M(D) - \nu^2 I]v_\pm^R$,
   and $B(D)w_\pm^{s,p,R}(x',0) = 0$;

2. $D^\beta w_\pm = \hat{w}_\pm = 0(|x|^{-1-\kappa}), \ \kappa \in (1/2, 1), \ |\beta| \leq 2$,
   $\hat{w}_\pm^s = g_{p,s}^\pm(x; \nu)[h(\omega)S_1^\pm(\omega; \nu) + v(\omega)S_2^\pm(\omega; \nu)]$,
   $\hat{w}_\pm^p = g_{p}^\pm(x; \nu)^t \omega P^\pm(\omega; \nu)$,
   where $g_{p,s}^\pm(x; \nu) = (4\pi|x|)^{-1} \exp(\pm i\nu|x|/c_{p,s})$, and $S_1, S_2, P$ are bounded scalar functions of $\omega, \nu$;

3. $D^\beta w_\pm^R(x; \nu) = c(\nu)\gamma_\pm(x)|x|^{-1/2} + R^\pm(x; \nu), \ |\beta| \leq 2$,
   $\gamma_\pm(x) = 
   \exp(\pm i\nu|x'|/c_R_0)[\gamma_1(\pm \alpha) \exp(-a_1 x_3) + \gamma_2(\pm \alpha) \exp(-ia_2 x_3)]$,
   $a_1, a_2 > 0, \ \gamma_1, \gamma_2$ smooth functions of $\alpha = x'/|x'|$, 

95
\[ \pm \int \gamma(x) t \mathcal{A}(\alpha, 0) E_0^{-1} A(D) \gamma(x) dx_3 = \text{const} > 0, \]

\[ |R^\pm(x; \nu)| \leq h(x_3)|x'|^{-1-\kappa'}, \kappa' \in (0, 1/2), h \in S(\mathbb{R}^+). \]

**Remark 9.1** It is clear that the solutions \( v^\pm(x; \nu) \) of Theorem 8.1 are contained in \( \mathcal{R}^\nu_\pm \).

**Theorem 9.2** Let \( v^\pm \in \mathcal{R}^\nu_\pm \) be solutions of

\[ [M(D) - \nu^2 I] v^\pm = f \epsilon D(\mathbb{R}^3). \]

Then \( v^\pm \) are the only solutions in \( \mathcal{R}^\nu_\pm \).

**Proof.** We suppose that \( w^\pm \in \mathcal{R}^\nu_\pm \) are solutions of 

\[ [M(D) - \nu^2 I] w^\pm = 0. \]

Then by property 1) of the class \( \mathcal{R}^\nu_\pm \) \([M(D) - \nu^2 I](w^\pm + w^\pm_s) = 0\) and \([M(D) - \nu^2 I] w^\pm_R = 0\) individually. To simplify the notation we drop the indices \( \pm \).

For \( 0 < A \in \mathbb{R} \) let \( B_A = \{ x \in \mathbb{R}^3_+ : |x| \leq A \} \). Denoting by \( dS \) the element of surface area of \( S^2 \), because of the boundary condition \( B(D)w_{p,s}(w', 0) = \mathcal{A}(n) E_0^{-1} A(D) w_{p,s}(x', 0) = 0, n = (0, 0, 1) \), we have

\[ 0 = \langle (w_p + w_s), [M(D) - \nu^2 I](w_p + w_s) \rangle_{B_A} - \]

\[ \langle [M(D) - \nu^2 I](w_p + w_s), (w_p + w_s) \rangle \]

\[ = -i \int_{\{|x|=A\} \cap \mathbb{R}^3_+} \tau (w_p + w_s) \mathcal{A}(\omega) E_0^{-1} A(D)(w_p + w_s) A^2 dS \]

96
\[ = -i \int_{\{|x| = A\} \cap \mathbb{R}^3_+} \hat{t}(\hat{w}_p + \hat{w}_s) t A(\omega) E_0^{-1} A(D)(\hat{w}_p + \hat{w}_s) A^2 dS + 0(A^{-\kappa}) \]

\[ = \pm \nu \int_{\{|x| = A\} \cap \mathbb{R}^3_+} \hat{t}(\hat{w}_p + \hat{w}_s)(c_p \hat{w}_p + c_s \hat{w}_s) A^2 dS \]

\[ + 0(|A|^{-\kappa}) \]

\[ = \pm \nu \int_{S^2_+} \{c_s|h(\omega)|^2|S_1(\omega)|^2 + c_s|v(\omega)|^2|S_2(\omega)|^2 + c_p|P(\omega)|^2\} dS + 0(A^{-\kappa}). \]

which implies \((A \to \infty)\) that \(0 = S_1(\omega) = S_2(\omega) = P(\omega)\) for a.e. \(\omega \in S^2_+\), and so \(w_{s,p} \in \mathcal{H}\).

Further, let \(C^L_A = \{x \in \mathbb{R}^3_+: |x'| \leq A, x_3 \leq L\} : \)

\[ 0 = \langle w_R, [M(D) - \nu^2 I]w_R \rangle_{C^L_A} - \langle [M(D) - \nu^2 I]w_R, w_R \rangle_{C^L_A} = \]

\[ -i \int_{|x'| \leq A} \hat{t} w_R(x', L) B(D) w_R(x', L) \]

\[ - i \int_0^L \hat{t} w_R \ t A(\alpha, 0) E_0^{-1} A(D) w_R A d\phi dx_3. \]

Letting \(L \to \infty\) gives

\[ 0 = \int_{S^1} \int_0^\infty |c(\nu)|^2 \hat{t} \gamma(x) t A(\alpha, 0) E_0^{-1} A(D) \gamma(x) dx_3 d\phi + \]

97
Letting $A \to \infty$ and integrating on $x_3$, we have

$$0 = |c(\nu)|^2 \int \int_{S^1} \int_0^\infty \bar{c}(\nu) \bar{\gamma}(x) \mathcal{A}(\alpha, 0) E_0^{-1} \mathcal{A}(D) \gamma(x) dx_3 d\phi = \text{const} |c(\nu)|^2.$$

Hence, $w_{\pm} \in \mathcal{H}$ which implies that $\nu^2 > 0$ is an eigenvalue. Since this is impossible, $w_{\pm} \equiv 0$. The proof is complete.

10 The Principle of Limiting Amplitude

We now show that the unique solutions of (5.2) are the limits as $t \to \infty$ of $\exp(i\nu t)u(x, t)$, where $u(x, t)$ is the solution of (5.1) with $f \in \mathcal{D}(\mathbb{R}^3)$, $u(x, 0) = u_t(x, 0) = 0$. This constitutes the principle of limiting amplitude and is the justification for the assumption often made in the applied literature that $u(x, t) = \exp(-i\nu t)v(x)$ where $v(x)$ is a solution of (5.2).
From Theorem 4.2 and (4.22) by Duhamel’s principle

\[ u(x, t) = \int_0^t [U(t - \tau)F]_1(x) \exp(-i\nu \tau) d\tau, \quad F = \begin{bmatrix} 0 \\ f \end{bmatrix}, \]

\[ \exp(i\nu t)u(x, t) = \int_0^t [U(\tau)F]_1(x) \exp(i\nu \tau) d\tau = \]

\[ \sum_{j=\pm 1} \int_0^t \{ \Psi_{jp}^* \exp(ijc_p \cdot |t|) \Psi_{jp} F(x) \]

\[ + \Psi_{js}^* \exp(ijc_s \cdot |t|) \Psi_{js} F(x) \]

\[ + \Sigma_j^* \exp(iJR(\cdot t)\Sigma_j F(x)) \} \exp(i\nu \tau) d\tau \]

\[ \equiv v_p(x, t) + v_s(x, t) + v_r(x, t). \]

We consider, for example, \( v_p(x, t) \). It can be shown by integration by parts [16] that \( \Psi_p^*(c_p \cdot |t|^{-1}) \exp(\pm ic_p \cdot |t|) \Psi_p f(x) \in L_1 \), so that by (4.22) (taking \( \Psi_p = \Psi_p^+ \) to be specific)

\[ \lim_{t \to \infty} v_p(x, t) = \]

\[ = -i2^{-1} \int_{\mathbb{R}^3} \Psi_p(x, \eta)(c_p|\eta|^{-1}) \exp[i\tau(c_p|\eta| + \nu)] \Psi_p f(\eta) d\eta d\tau + \]

\[ + i2^{-1} \int_{\mathbb{R}^3} \Psi_p(x, \eta)(c_p|\eta|)^{-1} \exp[i\tau(-c_p|\eta| + \nu)] \Psi_p f(\eta) d\eta d\tau = \]

99
\[-i2^{-1}\lim_{\epsilon \to 0} \int_{\mathbb{R}^3}^{\infty} \int_{0}^{\infty} \Psi_p(x, \eta)(c_p|\eta|)^{-1} \exp[i\tau(c_p|\eta| + \nu + i\epsilon)] d\tau d\eta +
\]

\[
\Psi_p f(\eta) d\tau d\eta +
\]

\[-i2^{-1}\lim_{\epsilon \to 0} \int_{\mathbb{R}^3}^{\infty} \int_{0}^{\infty} \Psi_p(x, \eta)(c_p|\eta|)^{-1} \exp[i\tau(-c_p|\eta| + \nu + i\epsilon)] d\tau d\eta
\]

\[= \lim_{\epsilon \to 0} \int_{\mathbb{R}^3}^{\infty} \Psi_p(x, \eta)[c_p|\eta|^2 - (\nu \pm i\epsilon)]^{-1} \Psi_p f(\eta) d\eta
\]

\[= v_p^{\pm}(x; \nu),
\]

where \(v_p^{\pm}(x; \nu)\) is the function of §8. In an altogether similar way \(v_s(x, t) \to v_s^{\pm}(x; \nu)\) and \(v_R(x, t) \to v_R^{\pm}(x; \nu)\), which establishes the advertised principle.
Appendix

We verify (6.14). To this end, for $\alpha \in S^1$ we write

$$t \mathcal{A}(\alpha, 0) E_0^{-1} \mathcal{A}(D) =$$

$$t \mathcal{A}(\alpha, 0) E_0^{-1} \mathcal{A}(D', 0) + t \mathcal{A}(\alpha, 0) E_0^{-1} \mathcal{A}(0', D_3),$$

$$t \mathcal{A}(\alpha, 0) E_0^{-1} \mathcal{A}(D', 0) =
\begin{bmatrix}
\alpha_1 c_p^2 D_1 + \alpha_2 c_s^2 D_2 & \alpha_1 \lambda D_2 + \alpha_2 c_s^2 D_1 & 0 \\
\alpha_2 \lambda D_1 + \alpha_1 c_s^2 D_2 & \alpha_2 c_p^2 D_2 + \alpha_1 c_s^2 D_1 & 0 \\
0 & 0 & c_s^2 (\alpha_1 D_1 + \alpha_2 D_2)
\end{bmatrix},$$

$$t \mathcal{A}(\alpha, 0) E_0^{-1} \mathcal{A}(0', D_3) =
\begin{bmatrix}
0 & 0 & \alpha_1 \lambda D_3 \\
0 & 0 & \alpha_2 \lambda D_3 \\
\alpha_1 c_s^2 D_3 & \alpha_2 c_s^2 D_3 & 0
\end{bmatrix}.$$

We set $a = \nu(1 - n^2 R_0^2)^{1/2}/c_s R_0$, $b = \nu(1 - R_0^2)^{1/2}/c_s R_0$,

$$A \equiv A(\alpha) =
\begin{bmatrix}
\alpha_1 c_p^2 + \alpha_2 c_s^2 & \alpha_1 \alpha_2 (\lambda + c_s^2) & 0 \\
\alpha_1 \alpha_2 (\lambda + c_s^2) & \alpha_2 c_p^2 + \alpha_1 c_s^2 & 0 \\
0 & 0 & c_s^2
\end{bmatrix},$$

101
\[ B \equiv B(\alpha) = \begin{bmatrix} 0 & 0 & \alpha_1 \lambda \\ 0 & 0 & \alpha_2 \lambda \\ \alpha_1 c_s^2 & \alpha_2 c_s^2 & 0 \end{bmatrix}, \]

\[ \tilde{\gamma}_\pm(x) = \exp(\mp i\nu|x'|/c_sR_0)(c_sR_0/\nu)^{1/2}\gamma_\pm(x). \]

Then

\[ i\gamma_\pm(x)A(\alpha, 0)E_0^{-1}A(D)\gamma_\pm(x) = \pm i\tilde{\gamma}_\pm(x)A\tilde{\gamma}(x) + i\tilde{\gamma}(x)B\lambda_\pm(x), \]

\[ \lambda_\pm(x) = i(R_0^2 - 2)\pi(\pm\alpha)\exp(-ax_3) + 
+ 2\sqrt{(1 - R_0^2)}\sigma(\pm\alpha)\exp(-bx_3), \]

\[ i\tilde{\gamma}_\pm(x)A\tilde{\gamma}_\pm(x) = (R_0^2 - 2)(1 - n^2 R_0^2)^{-1} i\pi(\pm\alpha)A\pi(\pm\alpha)\exp(-2ax_3) + 
+ 2i i\sigma(\pm\alpha)A\sigma(\pm\alpha)\exp(-2bx_3) + 
+ 4(R_0^2 - 2)(1 - n^2 R_0^2)^{-1/2} Im i\sigma(\pm\alpha)A\pi(\pm\alpha)\exp[-(a + b)x_3]. \]

\[ = (R_0^2 - 2)^2[c_p^2(1 - n^2 R_0^2)^{-1} + c_s^2]\exp(-2ax_3) + 
+ 4[c_s^2 + c_p^2(1 - R_0^2)]\exp(-2bx_3) - 
+ 4(2 - R_0^2)(1 - n^2 R_0^2)^{-1/2}[c_p^2 \sqrt{(1 - R_0^2)} + c_s^2 \sqrt{(1 - n^2 R_0^2)}] 
\exp[-(a + b)x_3]. \]
\[ \tilde{\gamma}_\pm(x) B \lambda_\pm(x) = \]

\[ = \pm (c_s^2 - \lambda)[R_0^2 - 2^2 \exp(-2ax_s) + 4(1 - R_0^2) \exp(-2b x_3)] \pm \\
2(R_0^2 - 2) \exp[-(a + b)x_3]\{c_s^2(1 - R_0^2) - \lambda \sqrt{(1 - R_0^2)} \times \\
\sqrt{(1 - n^2 R_0^2)} + (1 - n^2 R_0^2)^{-1/2}\}. \]

Hence, with \( \tilde{a} = (c_s R_0/\nu)a, \tilde{b} = (c_s R_0/\nu)b \)

\[ \pm \int_0^\infty \tilde{\gamma}_\pm(x)^4 A(\alpha, 0) E_0^{-1} A(D) \tilde{\gamma}_\pm(x) dx_3 \]

\[ = (R_0^2 - 2)^2(2\tilde{a})^{-1}[c_s^2(1 - n^2 R_0^2)^{-1} + 2c_s^2 - \lambda] \]

\[ + 12c_s^2(1 - R_0^2)(2\tilde{b})^{-1} + 2c_s^2\tilde{b}^{-1} + \\
2(R_0^2 - 2)(\tilde{a} + \tilde{b})^{-1}\{\lambda \sqrt{(1 - R_0^2)}[(1 - n^2 R_0^2)^{-1/2} \\
- \sqrt{(1 - n^2 R_0^2)}] + c_s^2[4 \sqrt{(1 - R_0^2)} \\
(1 - n^2 R_0^2)^{-1/2}] + 4 - R_0^2\} \]

\[ > 2c_s^2(R_0^2 - 2)^2\tilde{a}^{-1} + 6c_s^2(1 - R_0^2)\tilde{b}^{-1} + 2c_s^2\tilde{b}^{-1} \]

\[ - 2(2 - R_0^2)[2c_s^2 + c_s^2(2 - R_0^2)](\tilde{a} + \tilde{b})^{-1} \]

\[ > 6c_s^2(1 - R_0^2)\tilde{b}^{-1} - 4c_s^2(2 - R_0^2)(\tilde{a} + \tilde{b})^{-1} + 2c_s^2\tilde{b}^{-1} \]

\[ > 6c_s^2(1 - R_0^2)\tilde{b}^{-1} + 2c_s^2\tilde{b}^{-1} - 2c_s^2\tilde{b}^{-1}(2 - R_0^2) = 4c_s^2(1 - R_0^2)\tilde{b}^{-1} \]

103
References


