Wave Equation on $\mathbb{R}_+$ with Dirichlet BC

We consider the initial value problem for the wave equation on $0 < x < \infty$

$$u_{tt}(x,t) = c^2 u_{xx}(x,t), \quad 0 < x < \infty, \quad t > 0$$

$$u(x,0) = f(x), \quad 0 < x < \infty,$$

$$u_t(x,0) = g(x), \quad 0 < x < \infty,$$

$$u(0,t) = 0$$

(1)

(2)

Just as for the whole line it can be shown that the solution can be expressed as

$$u(x,t) = \varphi(x - ct) + \psi(x + ct)$$

(3)

for some sufficiently smooth functions $\varphi$ and $\psi$. Recall that we obtained for $x - ct > 0$

$$\varphi(x - ct) = \frac{1}{2} f(x - ct) - \left( \frac{1}{2c} \int_{x_0}^{x - ct} g(s) \, ds + \frac{K}{2} \right)$$

and

$$\psi(x + ct) = \frac{1}{2} f(x + ct) + \left( \frac{1}{2c} \int_{x_0}^{x + ct} g(s) \, ds + \frac{K}{2} \right)$$

But for this problem we need to find

$$\varphi(x - ct) \text{ for } -\infty < x - ct < \infty$$

and

$$\psi(x + ct) \text{ for } 0 < x + ct < \infty.$$  

For $x - ct \geq 0$ (which also implies $x - ct > 0$ since $x > 0$ and $t > 0$) we obtain the same result as before. Namely,

$$u(x,t) = \frac{f(x - ct) + f(x + ct)}{2} + \frac{1}{2c} \int_{x - ct}^{x + ct} g(s) \, ds.$$  

(4)

But for $x - ct < 0$ we need a different formula for $\varphi(x - ct)$ since $f$ is only defined for values of its argument which are greater than or equal to zero. What bails us out here is the boundary condition at $x = 0$.

Applying the boundary condition at $x = 0$ to $u(x,t) = \varphi(x - ct) + \psi(x + ct)$ we have

$$0 = u(0,t) = \varphi(-ct) + \psi(+ct) \text{ which implies}$$

$$\varphi(-ct) = -\psi(ct).$$

That is, for any $p < 0$ we can define $\varphi(p) = -\psi(-p)$. 

1
Thus for $x - ct < 0$ we define

$$
\varphi(x - ct) = -\psi(-(x - ct)) = -\psi(ct - x) = -\frac{1}{2}f(ct - x) - \left(\frac{1}{2c} \int_{ct-x}^{\infty} g(s) \, ds + \frac{K}{2}\right).
$$

Next we add the terms together to obtain, for $x - ct < 0$,

$$
u(x, t) = \frac{f(x - ct) - f(ct - x)}{2} + \frac{1}{2c} \int_{ct-x}^{x+ct} g(s) \, ds.
$$

Finally we can write the solution as

$$
u(x, t) = \begin{cases}
\frac{f(x - ct) + f(x + ct)}{2} + \frac{1}{2c} \int_{ct-x}^{x+ct} g(s) \, ds, & x > ct \\
\frac{f(x - ct) - f(ct - x)}{2} + \frac{1}{2c} \int_{ct-x}^{x+ct} g(s) \, ds, & x < ct
\end{cases}.
$$

(5)

**Remark 1.**

1. Equation (5) says that for $x > ct$ the solution is exactly the same as D’Alembert’s solution for an infinite wave, while for $x - ct$, the solution is modified as a result of the wave reflecting from the boundary (notice also that the sign of the wave is reflected, i.e. it becomes negative) (see Example 1 below).

2. The solution would, of course, change if we changed the boundary condition at zero. For example we could impose a Neumann condition.

**Example 1.** We consider the problem (1) with the initial conditions

$$f(x) = \begin{cases}
1 & 4 < x < 5 \\
0 & \text{otherwise}
\end{cases}
$$

and $g(x) = 0$. In this case the solution (5) becomes

$$u(x, t) = \begin{cases}
\frac{1}{2} \begin{cases} 
1 & 4 < x + t < 5 \\
0 & \text{otherwise}
\end{cases} + \frac{1}{2} \begin{cases} 
1 & 4 < x - t < 5 \\
0 & \text{otherwise}
\end{cases}, & t < x \\
\frac{1}{2} \begin{cases} 
1 & 4 < x + t < 5 \\
0 & \text{otherwise}
\end{cases} - \frac{1}{2} \begin{cases} 
1 & 4 < -x + t < 5 \\
0 & \text{otherwise}
\end{cases}, & x < t
\end{cases}.
$$