Heat Equation with a Radiation Boundary Condition

\[ u_t(x,t) = \alpha^2 u_{xx}(x,t), \quad 0 < x < \ell, \quad t > 0 \]  
\[ u(0,t) = 0, \quad u_x(1,t) + hu(1,t) = 0 \]  
\[ u(x,0) = \varphi(x) \]  

1. Separate Variables

Look for simple solutions in the form

\[ u(x,t) = X(x)T(t). \]

Substituting into (1) and dividing both sides by \( X(x)T(t) \) gives

\[ \frac{\dot{T}(t)}{T(t)} = \alpha^2 \frac{X''(x)}{X(x)} \]

Since the left side is independent of \( x \) and the right side is independent of \( t \), it follows that the expression must be a constant:

\[ \frac{\dot{T}(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)} = \lambda. \]

(Here \( \dot{T} \) means the derivative of \( T \) with respect to \( t \) and \( X' \) means means the derivative of \( X \) with respect to \( x \).) We seek to find all possible constants \( \lambda \) and the corresponding nonzero functions \( X \) and \( T \).

We obtain

\[ X'' - \lambda X = 0, \quad \dot{T} - \alpha^2 \lambda T = 0. \]

The solution of the second equation is

\[ T(t) = Ce^{\alpha^2 \lambda t} \]  

where \( C \) is an arbitrary constant. Furthermore, the boundary conditions give

\[ X(0)T(t) = 0, \quad X'(1) + hX(1)T(t) = 0 \quad \text{for all } t. \]

Since \( T(t) \) is not identically zero we obtain the desired eigenvalue problem

\[ X''(x) - \lambda X(x) = 0, \quad X(0) = 0, \quad X'(1) + hX(1)T(t) = 0. \]  

2. Find Eigenvalues and Eigenvectors

The next main step is to find the eigenvalues and eigenfunctions from (3). There are, in general, three cases:

(a) If \( \lambda = 0 \) then \( X(x) = ax + b \) so applying the boundary conditions we get

\[ 0 = X(0) = b, \quad 0 = X'(1) + hX(1) = a(1 + h) \quad \Rightarrow a = 0 \quad \text{unless } h = -1. \]

We conclude that \( \lambda_0 = 0 \) is note an eigenvalue unless \( h = -1 \).
(b) If \( \lambda = \mu^2 > 0 \) then
\[
X(x) = a \cosh(\mu x) + b \sinh(\mu x)
\]
and
\[
X'(x) = a\mu \sinh(\mu x) + b\mu \cosh(\mu x).
\]
Applying the boundary conditions we have
\[
0 = X(0) = a\mu \Rightarrow a = 0
\]
and
\[
0 = X'(1) + hX(1) = b(\mu \cosh(\mu) + h \sinh(\mu)) \quad \text{for} \quad b \neq 0 \quad \Rightarrow \tanh(\mu) = \frac{-\mu}{h}.
\]
This case is a bit more complicated depending on whether \( h \) is positive or negative. \( h \) positive corresponds to heat flowing out of the rod so there are no positive eigenvalues. There are also no positive eigenvalues for \(-1 < h < 0\). But for \( h < -1 \) we see that there is a single positive eigenvalue.

Consider the following alternative argument: If \( X''(x) = \lambda X(x) \) then multiplying by \( X \) we have \( X(x)X''(x) = \lambda X(x)^2 \). Integrate this expression from \( x = 0 \) to \( x = \ell \), apply integration by parts on the right and use \( X(0) = 0 \) and \( X'(1) = -hX(1) \).

We have
\[
\lambda \int_0^1 X(x)^2 \, dx = \int_0^1 X(x)X''(x) \, dx = -\int_0^1 X'(x)^2 \, dx - hX(1)^2 \bigg|_0^\ell.
\]
We conclude that
\[
\lambda = -\frac{\int_0^\ell X'(x)^2 \, dx + hX(1)^2}{\int_0^\ell X(x)^2 \, dx}
\]
and we see that \( \lambda \) is negative unless \( h \) is a large negative number. This calculation does not, however, give any real idea large \( \lambda \) needs to be. From here on we consider only the case \( h > 0 \).

(c) Finally, consider \( \lambda = -\mu^2 \) so that
\[
X(x) = a \cos(\mu x) + b \sin(\mu x)
\]
and
\[
X'(x) = -a\mu \sin(\mu x) + b\mu \cos(\mu x).
\]
Applying the boundary conditions we have
\[ 0 = X(0) = a \Rightarrow a = 0 \quad 0 = X'(1) + hX(1) = b(\mu \cos(\mu) + h \sin(\mu)) \]

From this we conclude
\[ \tan(\mu) = -\frac{\mu}{h}. \]

By graphing the functions on the right and left on the same axis it is easy to see that there are infinitely many values \( \mu_n \) with
\[ \frac{\pi}{2} < \mu_1 < \pi, \quad \frac{3\pi}{2} < \mu_2 < 2\pi, \]
and in general
\[ \frac{(2n - 1)\pi}{2} < \mu_n < n\pi, \quad \text{and} \quad \mu_n \xrightarrow{n \to \infty} \frac{(2n - 1)\pi}{2}. \]

Thus we have eigenvalues and eigenfunctions
\[ \lambda_n = -\mu_n^2, \quad X_n(x) = \sin(\mu_n x), \quad n = 1, 2, 3, \ldots. \quad (4) \]

From (2) we also have the associated functions \( T_n(t) = e^{\mu^2 \lambda_n t} \).

3. **Write as a Formal Sum** From the above considerations we can conclude that for any integer \( N \) and constants \( \{a_n\}_{n=0}^N \)
\[ u_n(x, t) = \sum_{n=1}^N b_n T_n(t) X_n(x) = \sum_{n=1}^N b_n e^{\lambda_n t} \sin(\mu_n x). \]
satisfies the differential equation in (1) and the boundary conditions.

4. **Use Fourier Series to Find Coefficients** The only problem remaining is to somehow pick the constants \( b_n \) so that the initial condition \( u(x, 0) = \varphi(x) \) is satisfied. To do this we need a theory which is more general than Fourier series. This theory is called
Sturm-Liouville theory and we will discuss it a little bit later. The main thing is that it guarantees that, just as with Fourier series, we look for $u$ as an infinite sum

$$u(x, t) = \sum_{n=1}^{\infty} b_n e^{\alpha x^2 \lambda_n t} \sin (\mu_n x)$$

and we seek $\{b_n\}$ satisfying

$$\varphi(x) = u(x, 0) = \sum_{n=1}^{\infty} b_n \sin (\mu_n x).$$

We claim (see proof at the end of the notes)

$$\int_0^1 X_n(x) X_m(x) \, dx = \int_0^1 \sin(\mu_n x) \sin(\mu_m x) \, dx = 0 \quad n \neq m \quad (5)$$

and

$$\int_0^1 X_n^2(x) \, dx = \int_0^1 \sin^2(\mu_n x) \, dx = \frac{(1 + h \cos^2(\mu_n))}{2} \equiv \frac{1}{\kappa_n} \neq 0 \quad (6)$$

so

$$\kappa_n = \frac{2}{(1 + h^{-1} \cos^2(\mu_n))} \xrightarrow{n \to \infty} 2.$$ 

Just as we did in our formal study of Fourier series, to find $b_n$ multiply both sides of the formal series by $X_n(x)$ and integrate from 0 to 1:

$$\int_0^1 \varphi(x) \sin(\mu_n x) \, dx = \sum_{k=1}^{\infty} b_k \int_0^1 \sin(\mu_k x) \sin(\mu_n x) \, dx$$

$$= b_n \int_0^1 \sin^2(\mu_n x) \, dx = \frac{b_n}{\kappa_n}$$

Thus we find

$$b_n = \kappa_n \int_0^1 \varphi(x) \sin (\mu_n x) \, dx. \quad (7)$$

As an explicit example for the initial condition consider $\varphi(x) = x$. In this case (7) becomes

$$b_n = \kappa_n \int_0^1 x \sin (\mu_n x) \, dx = \kappa_n \int_0^1 x \frac{- \cos (\mu_n x)}{\mu_n} \, dx$$

$$= \kappa_n \left[ \frac{x - \cos (\mu_n x)}{\mu_n} \right]_0^1 - \int_0^1 \frac{- \cos (\mu_n x)}{\mu_n} \, dx$$
\[ \kappa_n \left[ - \cos(\mu_n) \frac{1}{\mu_n} + \int_0^1 \cos(\mu_n x) \frac{1}{\mu_n} \, dx \right] \]

\[ = \kappa_n \left[ - \cos(\mu_n) + \frac{\sin(\mu_n)}{\mu_n^2} \right] \]

\[ = \kappa_n (h + 1) \sin(\mu_n) \frac{1}{\mu_n^2} \]

where on the last step we have used

\[ - \cos(\mu_n) = \frac{h \sin(\mu_n)}{\mu_n} \]

which follows from

\[ \tan(\mu) = \frac{-\mu}{h} . \]

So finally we arrive at the solution

\[ u(x, t) = \sum_{k=1}^{\infty} b_n e^{\alpha^2 \lambda_n t} \sin(\mu_n x). \]  \hspace{1cm} (8)

**Proof of Orthogonality and Derivation \( \kappa_n \)**

First we obtain the desired formula for \( \kappa_n \).

\[ \kappa_n^{-1} = \int_0^1 \sin^2(\mu_n x) \, dx = \frac{1}{2} \int_0^1 (1 - \cos(2\mu_n x)) \, dx \]

\[ = \frac{1}{2} \left[ x - \frac{\sin(2\mu_n x)}{2\mu_n} \right] \bigg|_0^1 = \frac{1}{2} \left[ 1 - \frac{\sin(2\mu_n)}{2\mu_n} \right] \]

\[ = \frac{1}{2} \left[ 1 - \frac{\sin(\mu_n) \cos(\mu_n)}{\mu_n} \right] \]

\[ = \frac{1}{2} \left[ 1 + h^{-1} \cos^2(\mu_n) \right] \]

where on the last step we have used

\[ \frac{\sin(\mu_n)}{\mu_n} = -\frac{\cos(\mu_n)}{h} \]

which follows from

\[ \tan(\mu) = \frac{-\mu}{h} . \]
Now we show orthogonality, i.e.,

\[ \int_0^1 \sin(\mu_n x) \sin(\mu_m x) \, dx = 0 \quad \text{for} \quad n \neq m. \]

Recall that \( \sin(\mu_j x) = X_j \) and that

\[ X_j'' = \lambda_j X_j, \quad X_j(0) = 0, \quad X_j'(1) = -h X_j(1), \quad j = n, m \]

so that

\[ \lambda_n \int_0^1 X_n(x) X_m(x) \, dx = \int_0^1 X_n''(x) X_m(x) \, dx \]

\[ = - \int_0^1 X_n'(x) X_m'(x) \, dx + X_n'(x) X_m(x) \bigg|_0^1 \]

\[ = \int_0^1 X_n(x) X_m''(x) \, dx + [X_n'(x) X_m(x) - X_n(x) X_m'(x)] \bigg|_0^1 \]

\[ = \lambda_m \int_0^1 X_n(x) X_m(x) \, dx + h [X_n(1) X_m(1) - X_n(1) X_m(1)] \]

\[ = \lambda_m \int_0^1 X_n(x) X_m(x) \, dx \]

Therefore we can conclude

\[ (\lambda_n - \lambda_m) \int_0^1 X_n(x) X_m(x) \, dx = 0 \]

and since \( \lambda_n \neq \lambda_m \) for \( n \neq m \) we have

\[ \int_0^1 X_n(x) X_m(x) \, dx = 0. \]