Heat Equation with Conduction and Convection

We consider the heat equation on the interval \((0, 1)\) with two extra terms that correspond to heat conduction and convection.

\[
\begin{align*}
  u_t (x, t) &= k (u_{xx}(x, t) - 2au(x, t)x + bu(x, t)), \quad 0 < x < \ell, \quad t > 0 \quad (1) \\
  u(0, t) &= 0, \quad (2) \\
  u(a, t) &= 0, \quad (3) \\
  u(x, 0) &= \varphi(x). \quad (4)
\end{align*}
\]

There are many different ways to approach this problem and one would be to apply separation of variables directly. The disadvantage to this is that one gets a more complicated ode for \(X(x)\) and there is a more difficult analysis of the eigenvalues and eigenvectors.

We will take a different approach which allows us to use our earlier work after a change of dependent variables. So to this end let us define \(v(x, t)\) via

\[
  u(x, t) = e^{ax+\beta t}v(x, t), \quad \beta = k(b - a^2). \quad (5)
\]

Thus we have

\[
v(x, t) = e^{-(ax+\beta t)}u(x, t)
\]

and we can compute

\[
  v_t - kv_{xx} = e^{-(ax+\beta t)} (-\beta u + u_t) - k \left[ e^{-(ax+\beta t)} (-au + u_x) \right]_x \\
  = e^{-(ax+\beta t)} \left\{ (-\beta u + u_t) - k \left[ -a(-au + u_x) + (-au_x + u_{xx}) \right] \right\} \\
  = e^{-(ax+\beta t)} \left[ u_t - k(u_{xx} - 2au_x + a^2 u) + \beta u \right] \\
  = e^{-(ax+\beta t)} \left[ u_t - k(u_{xx} - 2au_x + a^2 u + a^2 u + (b - a^2)u) \right] \\
  = e^{-(ax+\beta t)} \left[ u_t - k(u_{xx} - 2au_x + a^2 u + bu) \right] = 0.
\]

Furthermore

\[
  v(0, t) = e^{-\beta t}u(0, t) = 0, \quad v(1, t) = e^{-a-\beta t}u(1, t) = 0
\]

and

\[
  v(x, 0) = e^{-ax}u(x, 0) = e^{-ax}\varphi(x).
\]

Therefore, \(v(x, t)\) is the solution of

\[
  v_t = kv_{xx} \\
  v(0, t) = 0, \quad v(1, t) = 0 \\
  v(x, 0) = e^{-ax}\varphi(x).
\]

With \(\lambda_n = -(n\pi)^2\) the solution to this problem is

\[
  v(x, t) = \sum_{n=1}^{\infty} b_n e^{\lambda_n t} \sin(n\pi x) \quad \text{with} \quad b_n = 2 \int_0^1 e^{-ax}\varphi(x) \sin(n\pi x) \, dx.
\]

Finally our solution to (1)-(4) can be written as

\[
  u(x, t) = e^{ax+\beta t} \sum_{n=1}^{\infty} b_n e^{\lambda_n t} \sin(n\pi x).
\]