Heat Equation Neumann Boundary Conditions

\[ u_t(x, t) = u_{xx}(x, t), \quad 0 < x < \ell, \quad t > 0 \]  
\[ u_x(0, t) = 0, \quad u_x(\ell, t) = 0 \]  
\[ u(x, 0) = \varphi(x) \]  

1. Separate Variables

Look for simple solutions in the form

\[ u(x, t) = X(x)T(t). \]

Substituting into (1) and dividing both sides by \( X(x)T(t) \) gives

\[ \frac{\dot{T}(t)}{T(t)} = \frac{X''(x)}{X(x)} \]

Since the left side is independent of \( x \) and the right side is independent of \( t \), it follows that the expression must be a constant:

\[ \frac{\dot{T}(t)}{T(t)} = \frac{X''(x)}{X(x)} = \lambda. \]  

(Here \( \dot{T} \) means the derivative of \( T \) with respect to \( t \) and \( X' \) means means the derivative of \( X \) with respect to \( x \).) We seek to find all possible constants \( \lambda \) and the corresponding nonzero functions \( X \) and \( T \).

We obtain

\[ X'' - \lambda X = 0, \quad \dot{T} - \lambda T = 0. \]

The solution of the second equation is

\[ T(t) = Ce^{\lambda t} \]  

where \( C \) is an arbitrary constant. Furthermore, the boundary conditions give

\[ X'(0)T(t) = 0, \quad X'(\ell)T(t) = 0 \quad \text{for all } t. \]

Since \( T(t) \) is not identically zero we obtain the desired eigenvalue problem

\[ X''(x) - \lambda X(x) = 0, \quad X'(0) = 0, \quad X'(\ell) = 0. \]  

2. Find Eigenvalues and Eigenvectors

The next main step is to find the eigenvalues and eigenfunctions from (3). There are, in general, three cases:

(a) If \( \lambda = 0 \) then \( X(x) = ax + b \) and \( X'(x) = a \) so applying the boundary conditions we get

\[ 0 = X'(0) = a, \quad 0 = X'(\ell) = a \quad \Rightarrow a = 0. \]

Notice that \( b \) is still an arbitrary constant. We conclude that \( \lambda_0 = 0 \) is an eigenvalue with eigenfunction \( \varphi_0(x) = 1. \)
(b) If $\lambda = \mu^2 > 0$ then
\[ X(x) = a \cosh(\mu x) + b \sinh(\mu x) \]
and
\[ X'(x) = a\mu \sinh(\mu x) + b\mu \cosh(\mu x). \]
Applying the boundary conditions we have
\[ 0 = X'(0) = b\mu \Rightarrow b = 0 \quad 0 = X'(\ell) = a\mu \sinh(\mu \ell) \Rightarrow a = 0. \]
Therefore, there are no positive eigenvalues.
Consider the following alternative argument: If $X''(x) = \lambda X(x)$ then multiplying by $X$ we have $X(x)X''(x) = \lambda X(x)^2$. Integrate this expression from $x = 0$ to $x = \ell$. We have
\[ \lambda \int_0^\ell X(x)^2 \, dx = \int_0^\ell X(x)X''(x) \, dx = -\int_0^\ell X'(x)^2 \, dx + X(x)X'(x) \bigg|_0^\ell. \]
Since $X'(0) = X'(\ell) = 0$ we conclude
\[ \lambda = -\frac{\int_0^\ell X'(x)^2 \, dx}{\int_0^\ell X(x)^2 \, dx} \]
and we see that $\lambda$ must be less than or equal to zero (zero only if $X' = 0$).

(c) So, finally, consider $\lambda = -\mu^2$ so that
\[ X(x) = a \cos(\mu x) + b \sin(\mu x) \]
and
\[ X'(x) = -a\mu \sin(\mu x) + b\mu \cos(\mu x). \]
Applying the boundary conditions we have
\[ 0 = X'(0) = b\mu \Rightarrow b = 0 \quad 0 = X'(\ell) = -a\mu \sin(\mu \ell). \]
From this we conclude $\sin(\mu \ell) = 0$ which implies $\mu = \frac{n\pi}{\ell}$ and therefore
\[ \lambda_n = -\mu_n^2 = -\left(\frac{n\pi}{\ell}\right)^2, \quad X_n(x) = \cos(\mu_n x), \quad n = 1, 2, \ldots. \] (4)
From (2) we also have the associated functions $T_n(t) = e^{\lambda_n t}$.

3. Write Formal Infinite Sum From the above considerations we can conclude that for any integer $N$ and constants $\{a_n\}_{n=0}^N$
\[ u_n(x, t) = a_0 + \sum_{n=1}^N a_n T_n(t) X_n(x) = a_0 + \sum_{n=1}^N a_n e^{\lambda_n t} \cos \left(\frac{n\pi x}{\ell}\right). \]
satisfies the differential equation in (1) and the boundary conditions.
4. **Use Fourier Series to Find Coefficients** The only problem remaining is to somehow pick the constants $a_n$ so that the initial condition $u(x, 0) = \varphi(x)$ is satisfied. To do this we consider what we learned from Fourier series. In particular we look for $u$ as an infinite sum

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n e^{\lambda_n t} \cos \left( \frac{n\pi x}{\ell} \right)$$

and we try to find $\{a_n\}$ satisfying

$$\varphi(x) = u(x,0) = a_0 + \sum_{n=1}^{\infty} a_n \cos \left( \frac{n\pi x}{\ell} \right).$$

But this nothing more than a Cosine expansion of the function $\varphi$ on the interval $(0, \ell)$.

Our work on Fourier series showed us that

$$a_0 = \frac{1}{\ell} \int_0^\ell \varphi(x) \, dx, \quad a_n = \frac{2}{\ell} \int_0^\ell \varphi(x) \cos \left( \frac{n\pi x}{\ell} \right) \, dx.$$  \hspace{1cm} (5)

As an explicit example for the initial condition consider $\ell = 1$ and $\varphi(x) = x(1-x)$. In this case (5) becomes

$$a_0 = \int_0^1 \varphi(x) \, dx, \quad a_n = 2 \int_0^1 \varphi(x) \cos (n\pi x) \, dx.$$

We have

$$a_0 = \int_0^1 \varphi(x) \, dx = \int_0^1 x(1-x) \, dx$$

$$= \left[ \frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{1}{6},$$

and

$$a_n = 2 \int_0^1 \varphi(x) \cos (n\pi x) \, dx = 2 \int_0^1 x(1-x) \cos (n\pi x) \, dx$$

$$= 2 \int_0^1 (1-x) \left( \frac{\sin(n\pi x)}{n\pi} \right)' \, dx$$

$$= 2 \left[ (1-x) \frac{\sin(n\pi x)}{n\pi} \right]_0^1 - \int_0^1 (1-2x) \frac{\sin(n\pi x)}{n\pi} \, dx$$

$$= \frac{2}{n\pi} \int_0^1 (1-2x) \left( \frac{\cos(n\pi x)}{n\pi} \right)' \, dx$$

$$= \frac{2}{n\pi} \left[ (1-2x) \frac{\cos(n\pi x)}{n\pi} \right]_0^1 - \int_0^1 (-2) \frac{\cos(n\pi x)}{n\pi} \, dx$$
\[
\begin{align*}
\frac{2}{n\pi} \left[ -\cos \left( \frac{n\pi}{n}\right) - \frac{1}{n\pi} \right] \\
= \frac{-2}{(n\pi)^2}((-1)^n + 1) = \begin{cases} \\
\frac{-4}{(n\pi)^2}, & n \text{ even} \\
0, & n \text{ odd} \\
\end{cases}
\end{align*}
\]

In order to eliminate the odd terms in the expansion we introduce a new index, \( k \) by \( n = 2k \) where \( k = 1, 2, \ldots \). So finally we arrive at the solution

\[
u(x, t) = \frac{1}{6} + \frac{1}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2} e^{-4k^2\pi^2t} \cos(2k\pi x). \tag{6}
\]

Notice that as \( t \to \infty \) the infinite sum converges to zero uniformly in \( x \). Indeed,

\[
\left| \sum_{k=1}^{\infty} \frac{1}{k^2} e^{-4k^2\pi^2t} \cos(2k\pi x) \right| \leq e^{-4\pi^2t} \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} e^{-4\pi^2t}.
\]

So the solution converges to a nonzero steady state temperature which is exactly the average value of the initial temperature distribution.

\[
\lim_{t \to \infty} u(x, t) = \frac{1}{6} = \int_0^1 \varphi(x) \, dx.
\]