Heat Equation Dirichlet Boundary Conditions

\[ u_t(x, t) = k u_{xx}(x, t), \quad 0 < x < \ell, \quad t > 0 \tag{1} \]
\[ u(0, t) = 0, \quad u(\ell, t) = 0 \]
\[ u(x, 0) = \varphi(x) \]

1. Separate Variables

Look for simple solutions in the form
\[ u(x, t) = X(x)T(t). \]
Substituting into (1) and dividing both sides by \(X(x)T(t)\) gives
\[ \frac{\dot{T}(t)}{kT(t)} = \frac{X''(x)}{X(x)} \]
Since the left side is independent of \(x\) and the right side is independent of \(t\), it follows that the expression must be a constant:
\[ \frac{\dot{T}(t)}{kT(t)} = \frac{X''(x)}{X(x)} = \lambda. \]
(Here \(\dot{T}\) means the derivative of \(T\) with respect to \(t\) and \(X'\) means means the derivative of \(X\) with respect to \(x\).) We seek to find all possible constants \(\lambda\) and the corresponding nonzero functions \(X\) and \(T\).
We obtain
\[ X'' - \lambda X = 0, \quad \dot{T} - k\lambda T = 0. \]
The solution of the second equation is
\[ T(t) = C e^{k\lambda t} \tag{2} \]
where \(C\) is an arbitrary constant. Furthermore, the boundary conditions give
\[ X(0)T(t) = 0, \quad X(\ell)T(t) = 0 \quad \text{for all } t. \]
Since \(T(t)\) is not identically zero we obtain the desired eigenvalue problem
\[ X''(x) - \lambda X(x) = 0, \quad X(0) = 0, \quad X(\ell) = 0. \tag{3} \]

2. Find Eigenvalues and Eigenvectors

The next main step is to find the eigenvalues and eigenfunctions from (3). There are, in general, three cases:

(a) If \(\lambda = 0\) then \(X(x) = ax + b\) so applying the boundary conditions we get
\[ 0 = X(0) = b, \quad 0 = X(\ell) = a\ell \quad \Rightarrow \quad a = b = 0. \]
Zero is not an eigenvalue.
If $\lambda = \mu^2 > 0$ then 

\[ X(x) = a \cosh(\mu x) + b \sinh(\mu x). \]

Applying the boundary conditions we have 

\[ 0 = X(0) = a \Rightarrow a = 0 \quad 0 = X(\ell) = b \sinh(\mu \ell) \Rightarrow b = 0. \]

Therefore, there are no positive eigenvalues.

Consider the following alternative argument: If $X''(x) = \lambda X(x)$ then multiplying by $X$ we have $X(x)X''(x) = \lambda X(x)^2$. Integrate this expression from $x = 0$ to $x = \ell$. We have

\[ \lambda \int_0^{\ell} X(x)^2 \, dx = \int_0^{\ell} X(x)X''(x) \, dx = -\int_0^{\ell} X'(x)^2 \, dx + X(x)X'(x) \bigg|_0^{\ell}. \]

Since $X(0) = X(\ell) = 0$ we conclude

\[ \lambda = -\frac{\int_0^{\ell} X'(x)^2 \, dx}{\int_0^{\ell} X(x)^2 \, dx} \]

and we see that $\lambda$ must be less than or equal to zero.

(c) So, finally, consider $\lambda = -\mu^2$ so that

\[ X(x) = a \cos(\mu x) + b \sin(\mu x). \]

Applying the boundary conditions we have

\[ 0 = X(0) = a \Rightarrow a = 0 \quad 0 = X(\ell) = b \sin(\mu \ell). \]

From this we conclude $\sin(\mu \ell) = 0$ which implies

\[ \mu = \frac{n\pi}{\ell} \]

and therefore

\[ \lambda_n = -\mu_n^2 = -\left(\frac{n\pi}{\ell}\right)^2, \quad X_n(x) = \sin(\mu_n x), \quad n = 1, 2, \ldots. \quad (4) \]

From (2) we also have the associated functions $T_n(t) = e^{k\lambda_n t}$.

3. **Write Formal Sum** From the above considerations we can conclude that for any integer $N$ and constants $\{b_n\}_{n=0}^N$

\[ u_n(x, t) = \sum_{n=1}^N b_n T_n(t) X_n(x) = \sum_{n=1}^N b_n e^{k\lambda_n t} \sin \left(\frac{n\pi x}{\ell}\right). \]

satisfies the differential equation in (1) and the boundary conditions.
4. **Use Fourier Series to Find Coefficients**

The only problem remaining is to somehow pick the constants \(a_n\) so that the initial condition \(u(x, 0) = \varphi(x)\) is satisfied. To do this we consider what we learned from Fourier series. In particular we look for \(u\) as an infinite sum

\[
    u(x, t) = \sum_{n=1}^{\infty} b_n e^{k\lambda_n t} \sin \left( \frac{n\pi x}{\ell} \right)
\]

and we try to find \(\{b_n\}\) satisfying

\[
    \varphi(x) = u(x, 0) = \sum_{n=1}^{\infty} b_n \sin \left( \frac{n\pi x}{\ell} \right).
\]

But this nothing more than a Sine expansion of the function \(\varphi\) on the interval \((0, \ell)\).

\[
    b_n = 2 \ell \int_0^\ell \varphi(x) \sin \left( \frac{n\pi x}{\ell} \right) \, dx /.
\]

(5)

As an explicit example for the initial condition consider \(\ell = 1, k = 1/10\) and \(\varphi(x) = x(1-x)\).

Let us recall that \(\mu_n = \left( \frac{n\pi}{\ell} \right)\) which in this case reduces to \(n\pi\).

\[
    b_n = 2 \int_0^1 x(1-x) \sin (n\pi x) \, dx
\]

\[
    = 2 \int_0^1 x(1-x) \left( -\cos \left( \frac{n\pi x}{n\pi} \right) \right)' \, dx
\]

\[
    = \frac{2}{n\pi} \left[ -x(1-x) \frac{\cos(n\pi x)}{n\pi} \right]_0^1 + \int_0^1 (1-2x) \frac{\cos(n\pi x)}{\mu_n} \, dx
\]

\[
    = \frac{2}{n\pi} \int_0^1 (1-2x) \left( \frac{\sin(n\pi x)}{n\pi} \right)' \, dx
\]

\[
    = \frac{2}{n\pi} \left[ (1-2x) \frac{\sin(n\pi x)}{n\pi} \right]_0^1 + \int_0^1 (-2) \frac{\sin(n\pi x)}{n\pi} \, dx
\]

\[
    = \frac{-4}{(n\pi)^2} \int_0^1 \sin(n\pi x) \, dx
\]

\[
    = \frac{-4}{(n\pi)^2} \left[ -\frac{\cos(n\pi x)}{n\pi} \right]_0^1
\]

\[
    = \frac{4 \left[ (-1)^n - 1 \right]}{(n\pi)^3}
\]

We arrive at the solution

\[
    u(x, t) = \frac{4}{\pi^3} \sum_{n=1}^{\infty} \frac{[(-1)^n - 1]}{n^3} e^{-n^2\pi^2/10t} \sin (n\pi x).
\]

(6)