Heat Equation Dirichlet-Neumann Boundary Conditions

\[ u_t(x,t) = u_{xx}(x,t), \quad 0 < x < \ell, \quad t > 0 \]  
\[ u(0,t) = 0, \quad u_x(\ell,t) = 0 \]  
\[ u(x,0) = \varphi(x) \]  

1. Separate Variables  
Look for simple solutions in the form  
\[ u(x,t) = X(x)T(t). \]  
Substituting into (1) and dividing both sides by \( X(x)T(t) \) gives  
\[ \frac{\dot{T}(t)}{T(t)} \frac{T(t)}{X(x)} = \frac{X''(x)}{X(x)} = \lambda. \]

(Here \( \dot{T} \) means the derivative of \( T \) with respect to \( t \) and \( X' \) means means the derivative of \( X \) with respect to \( x \).) We seek to find all possible constants \( \lambda \) and the corresponding non-zero functions \( X \) and \( T \).  
We obtain  
\[ X'' - \lambda X = 0, \quad \dot{T} - \lambda T = 0. \]

The solution of the second equation is  
\[ T(t) = Ce^{\lambda t} \]  
where \( C \) is an arbitrary constant. Furthermore, the boundary conditions give  
\[ X(0)T(t) = 0, \quad X'(\ell)T(t) = 0 \quad \text{for all } t. \]

Since \( T(t) \) is not identically zero we obtain the desired eigenvalue problem  
\[ X''(x) - \lambda X(x) = 0, \quad X(0) = 0, \quad X'(\ell) = 0. \]  

2. Find Eigenvalues and Eigenvectors  
The next main step is to find the eigenvalues and eigenfunctions from (3). There are, in general, three cases:  
(a) If \( \lambda = 0 \) then \( X(x) = ax + b \) so applying the boundary conditions we get  
\[ 0 = X(0) = b, \quad 0 = X'(\ell) = a \quad \Rightarrow a = b = 0. \]

Zero is not an eigenvalue.
(b) If $\lambda = \mu^2 > 0$ then
\[ X(x) = a \cosh(\mu x) + b \sinh(\mu x) \]
and
\[ X'(x) = a \mu \sinh(\mu x) + b \mu \cosh(\mu x). \]
Applying the boundary conditions we have
\[ 0 = X'(0) = a \mu \Rightarrow a = 0 \quad 0 = X'(\ell) = b \mu \cosh(\mu \ell) \Rightarrow b = 0. \]
Therefore, there are no positive eigenvalues.
Consider the following alternative argument: If $X''(x) = \lambda X(x)$ then multiplying by $X$ we have $X(x)X''(x) = \lambda X(x)^2$. Integrate this expression from $x = 0$ to $x = \ell$. We have
\[ \lambda \int_0^\ell X(x)^2 \, dx = \int_0^\ell X(x)X''(x) \, dx = -\int_0^\ell X'(x)^2 \, dx + X(x)X'(x) \bigg|_0^\ell. \]
Since $X(0) = X'(\ell) = 0$ we conclude
\[ \lambda = -\frac{\int_0^\ell X'(x)^2 \, dx}{\int_0^\ell X(x)^2 \, dx} \]
and we see that $\lambda$ must be less than or equal to zero.
(c) So, finally, consider $\lambda = -\mu^2$ so that
\[ X(x) = a \cos(\mu x) + b \sin(\mu x) \]
and
\[ X'(x) = -a \mu \sin(\mu x) + b \mu \cos(\mu x). \]
Applying the boundary conditions we have
\[ 0 = X(0) = a \mu \Rightarrow a = 0 \quad 0 = X'(\ell) = b \mu \cos(\mu \ell). \]
From this we conclude $\cos(\mu \ell) = 0$ which implies
\[ \mu = \frac{(2n - 1)\pi}{2\ell} \]
and therefore
\[ \lambda_n = -\mu_n^2 = -\left(\frac{(2n - 1)\pi}{2\ell}\right)^2, \quad X_n(x) = \sin(\mu_n x), \quad n = 1, 2, \ldots . \quad (4) \]
From (2) we also have the associated functions $T_n(t) = e^{\lambda_n t}$.

3. Write Formal Infinite Sum From the above considerations we can conclude that for any integer $N$ and constants $\{b_n\}_{n=0}^N$
\[ u_n(x, t) = \sum_{n=1}^N b_n T_n(t) X_n(x) = \sum_{n=1}^N b_n e^{\lambda_n t} \sin \left(\frac{(2n - 1)\pi x}{2\ell}\right). \]
satisfies the differential equation in (1) and the boundary conditions.
4. **Use Fourier Series to Find Coefficients** The only problem remaining is to somehow pick the constants $a_n$ so that the initial condition $u(x,0) = \varphi(x)$ is satisfied. To do this we consider what we learned from Fourier series. In particular we look for $u$ as an infinite sum

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{\lambda_n t} \sin \left( \frac{(2n-1)\pi x}{2\ell} \right)$$

and we try to find $\{b_n\}$ satisfying

$$\varphi(x) = u(x,0) = \sum_{n=1}^{\infty} b_n \sin \left( \frac{(2n-1)\pi x}{2\ell} \right).$$

But this nothing more than a Sine type expansion of the function $\varphi$ on the interval $(0, \ell)$. Using

$$X_n(x) = \sin \left( \frac{(2n-1)\pi x}{2\ell} \right)$$

we have

$$\varphi(x) = \sum_{k=1}^{\infty} b_k X_k(x).$$

We proceed as usual by multiplying both sides by $X_n(x)$ and integrating from 0 to $\ell$ and using the orthogonality (described below in (6), (8)).

$$\int_0^{\ell} X_n(x)\varphi(x) \, dx = \sum_{k=1}^{\infty} b_k \int_0^{\ell} X_n(x)X_k(x) \, dx$$

which implies

$$b_n = \frac{2}{\ell} \int_0^{\ell} \varphi(x)X_n(x) \, dx. \quad (5)$$

**orthogonality:**

$$\int_0^{\ell} X_n(x)X_k(x) \, dx = \begin{cases} \frac{\ell}{2}, & n = k \\ 0, & n \neq k \end{cases}. \quad (6)$$

To see this recall that $X''_j = \lambda_j X_j$ and $X_j(0) = 0$, $X'_j(\ell) = 0$. First consider $n \neq k$ so $\lambda_n \neq \lambda_k$ and therefore

$$\lambda_n \int_0^{\ell} X_n(x)X_k(x) \, dx = \int_0^{\ell} X''_n(x)X_k(x) \, dx$$

$$= - \int_0^{\ell} X'_n(x)X'_k(x) \, dx + [X'_n(x)X_k(x)]_0^{\ell}$$

$$= \int_0^{\ell} X_n(x)X''_k(x) \, dx + [X'_n(x)X_k(x) - X_n(x)X'_k(x)]_0^{\ell}$$

$$= \lambda_k \int_0^{\ell} X_n(x)X_k(x) \, dx. \quad (7)$$
Therefore

\[(\lambda_n - \lambda_k) \int_0^\ell X_n(x)X_k(x) \, dx = 0 \Rightarrow \int_0^\ell X_n(x)X_k(x) \, dx = 0.\]

For \(n = k\) we have

\[\begin{align*}
\int_0^\ell X_n^2(x) \, dx &= \int_0^\ell \sin^2 \left( \frac{(2n - 1)\pi x}{2\ell} \right) \, dx \\
&= \frac{1}{2} \int_0^\ell \left[ 1 - \cos \left( \frac{(2n - 1)\pi x}{\ell} \right) \right] \, dx \\
&= \frac{\ell}{2} - \frac{1}{2} \left( \frac{\ell}{(2n - 1)\pi} \right) \sin \left( \frac{(2n - 1)\pi x}{\ell} \right) \bigg|_0^\ell \\
&= \frac{\ell}{2}. \quad (9)
\end{align*}\]

As an explicit example for the initial condition consider \(\varphi(x) = x\). Let us recall that \(\mu_n = \left( \frac{(2n - 1)\pi}{2\ell} \right)\)

\[b_n = \frac{2}{\ell} \int_0^\ell \varphi(x)X_n(x) \, dx = \frac{2}{\ell} \int_0^\ell x \sin(\mu_n x) \, dx\]

\[= \frac{2}{\ell} \int_0^\ell x \left( -\cos \left( \frac{\mu_n x}{\mu_n} \right) \right) ' \, dx\]

\[= \frac{2}{\ell} \left[ -x \frac{\cos(\mu_n x)}{\mu_n} \bigg|_0^\ell + \int_0^\ell \frac{\cos(\mu_n x)}{\mu_n} \, dx \right]\]

\[= \frac{2}{\ell} \left[ -\ell \frac{\cos(\mu_n \ell)}{\mu_n} + \frac{\sin(\mu_n x)}{\mu_n^2} \bigg|_0^\ell \right]\]

\[= \frac{2}{\ell} \left[ -\ell \frac{\cos((2n - 1)\pi/2)}{\mu_n} + \frac{\sin((2n - 1)\pi/2)}{\mu_n^2} \right]\]

\[= \frac{8\ell(-1)^{n+1}}{(2n - 1)^2\pi^2}\]

We arrive at the solution

\[u(x, t) = \frac{8\ell}{\pi^2} \sum_{n=1}^\infty \frac{(-1)^{n+1}}{(2n - 1)^2} e^{\lambda_n t} \sin(\mu_n x). \quad (10)\]