Math 4354, Assignment Number 5 Solutions

1.

\[ u_{tt}(x, t) = c^2 u_{xx}(x, t), \quad 0 < x < \pi, \quad t > 0 \]
\[ u(0, t) = 0, \quad u(\pi, t) = 0 \]
\[ u(x, 0) = x, \quad u_t(x, 0) = 0 \]

Solution:
Look for simple solutions in the form

\[ u(x, t) = X(x)T(t). \]

We obtain

\[ X'' - \lambda X = 0, \quad X(0) = X(\pi) = 0, \quad T'' - \lambda T = 0. \]

By now we know for Dirichlet conditions on the interval \((0, \pi)\) we get

\[ \lambda_n = -n^2, \quad X_n(x) = \sin(nx) \]

and

\[ T_n(t) = a_n \cos(nct) + b_n \sin(nct). \]

The only problem remaining is to somehow pick the constants \(a_n\) so that the initial conditions \(u(x, 0) = x\) and \(u_t(x, 0) = 0\) are satisfied. We look for \(u\) as an infinite sum

\[ u(x, t) = \sum_{n=1}^{\infty} (a_n \cos(nct) + b_n \sin(nct)) \sin(nx) \]

and we try to find \(\{a_n\}\) satisfying

\[ x = u(x, 0) = \sum_{n=1}^{\infty} a_n \sin(nx), \quad 0 = u_t(x, 0) = \sum_{n=1}^{\infty} nb_n \sin(nx). \]

The last equation implies that \(b_n = 0\) for all \(n\). But this nothing more than a Sine expansion of the function \(x\) on the interval \((0, \pi)\).

\[ a_n = \frac{2}{\pi} \int_0^{\pi} x \sin(nx) \, dx/ \]
\[ = \frac{2}{\pi} \int_0^{\pi} x \left(-\frac{\cos(nx)}{n}\right)' \, dx \]
\[ = \frac{2}{\pi} \left[ x \left(-\frac{\cos(nx)}{n}\right) \right]_0^{\pi} - \int_0^{\pi} x \left(-\frac{\cos(nx)}{n}\right)' \, dx \]
\[ = \frac{2(-1)^{n+1}}{n} \]

\[ u(x, t) = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cos(nct) \sin(nx). \]
\[ u_{tt}(x, t) = u_{xx}(x, t), \quad 0 < x < 1, \quad t > 0 \]
\[ u_x(0, t) = 0, \quad u_x(1, t) = 0 \]
\[ u(x, 0) = x^2, \quad u_t(x, 0) = 0 \]

**Solution:**

Look for simple solutions in the form
\[ u(x, t) = X(x)T(t). \]

We obtain
\[ X'' - \lambda X = 0, \quad X'(0) = X'(1) = 0, \quad T'' - \lambda T = 0. \]

By now we know for Neumann conditions on the interval \((0, \pi)\) we get
\[ \lambda_n = -(n\pi)^2, \quad X_n(x) = \sin(n\pi x) \]
and
\[ T_n(t) = a_n \cos(n\pi t) + b_n \sin(n\pi t). \]

The only problem remaining is to somehow pick the constants \(a_n\) so that the initial conditions \(u(x, 0) = x^2\) and \(u_t(x, 0) = 0\) are satisfied. Just as in the previous problems, since \(u_t(x, 0) = 0\), we have \(b_n = 0\) for all \(n\). So we have
\[ a_0 = \int_0^1 x^2 \, dx, \quad a_n = 2 \int_0^1 x^2 \cos(n\pi x) \, dx. \]

\[ a_0 = \frac{1}{\pi} \int_0^\pi \varphi(x) \, dx = \frac{1}{\pi} \int_0^1 x^2 \, dx \]
\[ = \left[ \frac{x^3}{3} \right]_0^1 = \frac{1}{3}. \]

and
\[ a_n = 2 \int_0^1 x^2 \cos(n\pi x) \, dx = 2 \int_0^1 x^2 \left( \frac{\sin(n\pi x)}{n\pi} \right) ' \, dx \]
\[ = 2 \left[ x^2 \frac{\sin(n\pi x)}{n\pi} \right]_0^1 - \int_0^1 2x \frac{\sin(n\pi x)}{n\pi} \, dx \]
\[ = -\frac{4}{n\pi} \int_0^1 x \sin(n\pi x) \, dx = -\frac{4}{n\pi} \int_0^1 x \left( \frac{-\cos(n\pi x)}{n\pi} \right) ' \, dx \]
\[ = -\frac{4}{n\pi} \left[ x \left( \frac{-\cos(n\pi x)}{n\pi} \right) \right]_0^1 + \int_0^1 \frac{\cos(n\pi x)}{n\pi} \, dx \]
\[ = \frac{4(-1)^n}{n^2\pi^2}. \]
\[
u(x, t) = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(n\pi t) \cos(n\pi x).
\]

3.

\[
u_{tt}(x, t) = \nu_{xx}(x, t), \quad 0 < x < \pi, \quad t > 0
\]
\[
u(0, t) = 0, \quad \nu_x(\pi, t) = 0
\]
\[
u(x, 0) = x, \quad \nu_t(x, 0) = 0
\]

**Solution:** Look for simple solutions in the form

\[
u(x, t) = X(x)T(t)
\]

from which we obtain

\[X''(x) - \lambda X(x) = 0, \quad X(0) = 0, \quad X'(\pi) = 0, \quad T'' - \lambda T = 0.
\]

Therefore

\[
\lambda_n = -\mu_n^2 = - \left( \frac{2n-1}{2} \right)^2, \quad X_n(x) = \sin(\mu_n x), \quad n = 1, 2, \ldots
\]

and

\[
T_n(t) = a_n \cos(nct) + b_n \sin(nct).
\]

We look for \(u\) as an infinite sum

\[
u(x, t) = \sum_{n=1}^{\infty} [a_n \cos(nct) + b_n \sin(nct)] \sin \left( \frac{(2n-1)x}{2} \right).
\]

Just as in the previous problems, since \(u_t(x, 0) = 0\), we have \(b_n = 0\) for all \(n\). So we try to find \(\{n_n\}\) satisfying

\[
x = u(x, 0) = \sum_{n=1}^{\infty} a_n \sin \left( \frac{(2n-1)x}{2} \right).
\]

For the initial condition \(\varphi(x) = x\). Recall that \(\mu_n = \left( \frac{2n-1}{2} \right)\)

\[
a_n = \frac{2}{\pi} \int_0^\pi x \sin(\mu_n x) \, dx = \frac{2}{\pi} \int_0^\pi x \left( -\frac{\cos(\mu_n^2 x)}{\mu_n^2} \right)' \, dx
\]

\[
= \frac{2}{\pi} \left[ -x \frac{\cos(\mu_n x)}{\mu_n} \right]_0^\pi + \int_0^\pi \frac{\cos(\mu_n x)}{\mu_n} \, dx
\]

\[
= \frac{2}{\pi} \left[ -\pi \frac{\cos(\mu_n \pi)}{\mu_n} + \frac{\sin(\mu_n \pi)}{\mu_n^2} \right]_0^\pi
\]

\[
= \frac{2}{\pi} \left[ -\pi \frac{\cos((2n-1)\pi/2)}{\mu_n} + \frac{\sin((2n-1)\pi/2)}{((2n-1)/2)^2} \right]
\]

\[
= \frac{8(-1)^{n+1}}{(2n-1)^2 \pi}
\]
\[ u(x, t) = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^3} \cos \left( \frac{(2n-1)t}{2} \right) \sin \left( \frac{(2n-1)x}{2} \right). \]

4. This problem is concerned with the heat equation on the whole line. Verify by direct differentiation that, for fixed parameters \( k \) and \( y \), the function

\[ k(x, t) = \frac{1}{\sqrt{4\pi kt}} e^{-(x-y)^2/(4t)} \]

satisfies the heat equation, i.e. show that \( k_t - k_{xx} = 0 \) for all \( t > 0 \) and all \( x \in \mathbb{R} \).

**Solution:** We compute the derivative \( k_t(x, t) \) using the product rule and chain rule:

\[
k_t(x, t) = \frac{1}{\sqrt{4k\pi}} \frac{d}{dt} \left( t^{-1/2} e^{-(x-y)^2/(4t)} \right)
= \frac{1}{\sqrt{4k\pi}} \left[ (-1/2) t^{-3/2} + \left( t^{-1/2} 4(x-y)^2 \right) \right] e^{-(x-y)^2/(4t)}
= \frac{1}{\sqrt{4k\pi}} e^{-(x-y)^2/(4t)} \left[ \frac{-1}{2t^{3/2}} + \frac{(x-y)^2}{4t^{5/2}} \right]
= - \frac{1}{\sqrt{4k\pi}} e^{-(x-y)^2/(4t)} \left[ \frac{2t - (x-y)^2}{4t^{5/2}} \right]
= - \frac{1}{\sqrt{4k\pi t}} e^{-(x-y)^2/(4t)} \left[ \frac{2t - (x-y)^2}{4t^2} \right]
\]

Next we compute \( k_x(x, t) \) using the chain rule

\[
k_x(x, t) = \frac{-(x-y)}{2\sqrt{2k\pi t}} e^{-(x-y)^2/(4t)}
\]

and \( k_{xx}(x, t) \) using the chain rule and product rule

\[
k_{xx}(x, t) = \frac{-1}{2t \sqrt{4k\pi t}} e^{-(x-y)^2/(4t)} + \frac{(x-y)^2}{4t^2 \sqrt{4k\pi t}} e^{-(x-y)^2/(4t)}
= - \frac{1}{\sqrt{4k\pi t}} e^{-(x-y)^2/(4t)} \left[ \frac{2t - (x-y)^2}{4t^2} \right]
\]

Thus we obtain

\[
k_t(x, t) - k_{xx}(x, t) = 0.
\]