Math 4354,  Assignment Number 4

Use D’Alembert’s formula to solve the following wave equation problems on \( \mathbb{R} \).

1. 

\[ u_{tt}(x, t) = 9u_{xx}(x, t), \quad -\infty < x < \infty \quad t > 0 \]  
\[ u(x, 0) = 1, \quad u_t(x, 0) = 0 \]  

**Solution:** We apply the D’Alembert formula for the solution

\[ u(x, t) = \frac{u(x + 3t, 0) + u(x - 3t, 0)}{2} = \frac{1 + 1}{2} = 1. \]

2. 

\[ u_{tt}(x, t) = 4u_{xx}(x, t), \quad -\infty < x < \infty \quad t > 0 \]  
\[ u(x, 0) = 0, \quad u_t(x, 0) = 1 \]  

**Solution:** We apply the D’Alembert formula for the solution

\[ u(x, t) = \frac{1}{4} \int_{x-2t}^{x+2t} u_t(s, 0) \, ds = \frac{1}{4} \int_{x-2t}^{x+2t} 1 \, ds = \frac{(x + 2t) - (x - 2t)}{4} = t. \]

3. 

\[ u_{tt}(x, t) = u_{xx}(x, t), \quad -\infty < x < \infty \quad t > 0 \]  
\[ u(x, 0) = \sin(x), \quad u_t(x, 0) = \cos(x) \]  

**Solution:** We apply the D’Alembert formula for the solution

\[ u(x, t) = \frac{\sin(x + t) + \sin(x - t)}{2} + \frac{1}{2} \int_{x-t}^{x+t} \cos(s) \, ds 
= \frac{\sin(x + t) + \sin(x - t)}{2} + \frac{\sin(x + t) - \sin(x - t)}{2} = \sin(x + t) \]

4. 

\[ u_{tt}(x, t) = u_{xx}(x, t), \quad -\infty < x < \infty \quad t > 0 \]  
\[ u(x, 0) = e^x, \quad u_t(x, 0) = \frac{1}{1 + x^2} \]  

We apply the D’Alembert formula for the solution

\[ u(x, t) = \frac{e^{x+t} + e^{x-t}}{2} + \frac{1}{2} \int_{x-t}^{x+t} \frac{1}{1 + s^2} \, ds 
= e^x \sinh(t/2) + \frac{1}{2} \left( \tan^{-1}(x + t) - \tan^{-1}(x - t) \right) \]
\[ u_{tt}(x, t) = 4u_{xx}(x, t), \quad -\infty < x < \infty \quad t > 0 \]
\[ u(x, 0) = 0, \quad u_t(x, 0) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| > 1 \end{cases} \]
\[ x + ct = -1 \quad x + ct = 1 \quad x - ct = -1 \quad x - ct = 1 \]

To do Problem 5 simply pick a point \((x, t)\) in each region and compute the integral
\[ \frac{1}{2c} \int_{x - ct}^{x + ct} g(s) \, ds \]

(a) If \((x, t)\) is in region \(R_1\) then \(x + ct < -1\) which means that the upper limit of integration is less than \(-1\) so \(g(s) = 0\) on \([x - ct, x + ct]\). Therefore
\[ u(x, t) = \frac{1}{2c} \int_{x - ct}^{x + ct} g(s) \, ds = 0, \quad \forall \ (x, t) \in R_1. \]

(b) If \((x, t)\) is in region \(R_2\) then \(-1 < x + ct < 1\) and \(x - ct < -1\) which means that on \([x - ct, x + ct]\) we have
\[ u(x, t) = \frac{1}{2c} \int_{x - ct}^{x + ct} g(s) \, ds = \frac{1}{2c} \int_{-1}^{x + ct} 1 \, ds = \frac{x + ct + 1}{2c}, \quad \forall \ (x, t) \in R_2. \]

(c) For \((x, t)\) in \(R_3\) then \(x + ct > 1\) and \(x - ct < -1\) so we have
\[ u(x, t) = \frac{1}{2c} \int_{x - ct}^{1} 1 \, ds = \frac{1}{c}, \quad \forall \ (x, t) \in R_3. \]

(d) For \((x, t)\) in \(R_4\) then \(x + ct > 1\) and \(-1 \leq x - ct \leq 1\) so we have
\[ u(x, t) = \frac{1}{2c} \int_{x - ct}^{1} 1 \, ds = \frac{1 - x + ct}{2c}, \quad \forall \ (x, t) \in R_4. \]

(e) For \((x, t)\) in \(R_5\) then \(x - ct > 1\) so we have
\[ u(x, t) = \frac{1}{2c} \int_{x - ct}^{x + ct} 0 \, ds = 0, \quad \forall \ (x, t) \in R_5. \]

(f) For \((x, t)\) in \(R_6\) then \(x + ct < 1\) and \(x - ct > 1\) so we have
\[ u(x, t) = \frac{1}{2c} \int_{x - ct}^{x + ct} 1 \, ds = \frac{(x + ct) - (x - ct)}{c} = t, \quad \forall \ (x, t) \in R_6. \]