Use D'alemberts formula to solve the following wave equation problems on $\mathbb{R}$.

1. 

$$u_{tt}(x, t) = 9u_{xx}(x, t), \quad -\infty < x < \infty \quad t > 0$$  \hspace{1cm} (1) 

$$u(x, 0) = 1, \quad u_t(x, 0) = 0$$

2. 

$$u_{tt}(x, t) = 4u_{xx}(x, t), \quad -\infty < x < \infty \quad t > 0$$  \hspace{1cm} (2) 

$$u(x, 0) = 0, \quad u_t(x, 0) = 1$$

3. 

$$u_{tt}(x, t) = u_{xx}(x, t), \quad -\infty < x < \infty \quad t > 0$$  \hspace{1cm} (3) 

$$u(x, 0) = \sin(x), \quad u_t(x, 0) = \cos(x)$$

4. 

$$u_{tt}(x, t) = u_{xx}(x, t), \quad -\infty < x < \infty \quad t > 0$$  \hspace{1cm} (4) 

$$u(x, 0) = e^x, \quad u_t(x, 0) = \frac{1}{1 + x^2}$$

5. 

$$u_{tt}(x, t) = 4u_{xx}(x, t), \quad -\infty < x < \infty \quad t > 0$$  \hspace{1cm} (5) 

$$u(x, 0) = 0, \quad u_t(x, 0) = \begin{cases} 
1, & |x| < 1 \\
0, & |x| > 1 
\end{cases}$$

In order to express the answer it is useful to use the Heaviside function notation

$$H(x) = \begin{cases} 
0, & x < 0 \\
1, & x > 0 
\end{cases}$$

So for example we can write

$$u(x, 0) = H(x + 1) - H(x - 1).$$
To do Problem 5 simply pick a point \((x, t)\) in each region and compute the integral

\[
\frac{1}{2c} \int_{x-ct}^{x+ct} g(s) \, ds
\]

As an example let us do the values for the regions \(R1\) and \(R2\).

(a) If \((x, t)\) is in region \(R1\) then \(x + ct < -1\) which means that the upper limit of integration is less than \(-1\) so \(g(s) = 0\) on \([x - ct, x + ct]\). Therefore

\[
u(x, t) = \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) \, ds = 0, \quad \forall \quad (x, t) \in R1.
\]

(b) If \((x, t)\) is in region \(R2\) then \(-1 < x + ct < 1\) and \(x - ct < -1\) which means that on \([x - ct, x + ct]\) we have

\[
u(x, t) = \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) \, ds = \frac{1}{2c} \int_{-1}^{x+ct} 1 \, ds = \frac{x + ct + 1}{2c}, \quad \forall \quad (x, t) \in R2.
\]