1. The Fourier series for \( f(x) = x^2 \) on \(-\pi \leq x \leq \pi\) is
\[
\frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx)
\] (1)
Find values of \( x \) and give a justification for using them along with the above information to show that
(a) \( \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \)

**Solution:**
First we note that since \( f(x) = x^2 \) on \(-\pi \leq x \leq \pi\) is a \( PC^{(1)} \) function which is also continuous we know that the Fourier series converges point at every value of \( x \), i.e.,
\[
x^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx).
\]
Setting \( x = \pi \) in (1) we arrive at
\[
\pi^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(n\pi)
\]
Now \( \cos(n\pi) = (-1)^n \) so we have
\[
\pi^2 - \frac{\pi^2}{3} = 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = 4 \sum_{n=1}^{\infty} \frac{1}{n^2}
\]
Finally
\[
\pi^2 - \frac{\pi^2}{3} = \frac{2\pi^2}{3}
\]
so we can divide by 4 to get the desired result.

(b) \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12} \)

**Solution:**
Just as in the previous part we know that
\[
x^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx).
\]
Setting $x = 0$ in (1) we arrive at

$$0 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(0)$$

which implies

$$-\frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

Finally multiply both sides by $(-1)$ to get the desired result.

2. Find the Fourier Cosine series expansion of $f(x) = \sin(x)$ on $0 \leq x \leq \pi$.

**Solution:** For the Fourier cosine series we have the formulas:

$$a_0 = \frac{1}{\pi} \int_{0}^{\pi} \sin(x) \, dx, \quad a_n = \frac{2}{\pi} \int_{0}^{\pi} \cos(nx) \sin(x) \, dx$$

Thus we obtain

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx = \frac{1}{\pi} \int_{0}^{\pi} \sin(x) \, dx = \frac{-[\cos(\pi) - \cos(0)]}{\pi} = \frac{2}{\pi},$$

and

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx = \frac{2}{\pi} \int_{0}^{\pi} \sin(x) \cos(nx) \, dx$$

$$= \frac{1}{\pi} \int_{0}^{\pi} [\sin((n + 1)x) - \sin((n - 1)x)] \, dx$$

$$= \frac{1}{\pi} \left[ \frac{-\cos((n + 1)x)}{n+1} + \frac{\cos((n - 1)x)}{n-1} \right]_{0}^{\pi}$$

$$= \frac{1}{\pi} \left[ \frac{(-1)^{n+1} - 1}{n+1} + \frac{(-1)^{n-1} - 1}{n-1} \right]$$

$$= \frac{[1 + (-1)^n]}{\pi} \left[ \frac{1}{n+1} - \frac{1}{n-1} \right]$$

$$= \frac{-2[1 + (-1)^n]}{(n^2 - 1)\pi}$$

We see that $a_n = 0$ for $n$ odd. Thus setting $n = 2k$ we can write the solution

$$\sin(x) = \frac{2}{\pi} - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\cos(2kx)}{(4k^2 - 1)}.$$  \hspace{1cm} (2)

You will note that the resulting series converges uniformly so we can differentiate and integrate the series termwise. Note, in addition, that

$$\cos(x) = \frac{d}{dx} \sin(x) = -\int_{\pi/2}^{x} \sin(s) \, ds.$$
We can use this to obtain two different Fourier Series expansions for \( \cos(x) \) which are valid for \( 0 < x < \pi \). In particular:

(a) Differentiate the series for \( f \).

If we differentiate (2) with respect to \( x \) we get

\[
\cos(x) = \frac{8}{\pi} \sum_{k=1}^{\infty} \frac{k \sin(2kx)}{(4k^2 - 1)}. \quad (3)
\]

(b) Integrate (as above) the series for \( f \),

\[
\cos(x) = - \int_{\pi/2}^{x} \sin(s) \, ds =
\]

\[
= - \int_{\pi/2}^{x} \frac{2}{\pi} \, ds + \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\int_{\pi/2}^{x} \cos(2ks) \, ds}{(4k^2 - 1)}
\]

\[
= - \frac{2}{\pi} \left[ x - \frac{\pi}{2} \right] + \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\sin(2ks)|_{\pi/2}^{x}}{(2k)(4k^2 - 1)}
\]

\[
= 1 - \frac{2}{\pi} x + \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\sin(2kx)}{(2k)(4k^2 - 1)}
\]

3. Bonus question (which means that it is worth extra credit and is not part of the actual assignment) try to reconcile the difference between the two series expansions for \( \cos(x) \) on \( 0 < x < \pi \). You might find the following formula useful:

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx) = \frac{x}{2}, \quad -\pi < x < \pi.
\]

In order to show that (3) and (4) are the same we need to show that

\[
\frac{8}{\pi} \sum_{k=1}^{\infty} \frac{k \sin(2kx)}{(4k^2 - 1)} = 1 - \frac{2x}{\pi} + \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{\sin(2kx)}{k(4k^2 - 1)}.
\]

Subtracting the sum on the right from both sides and combining the two infinite sums we arrive at the following (which needs to be true)

\[
\frac{1}{\pi} \sum_{k=1}^{\infty} \left( 8k - \frac{2}{k} \right) \frac{\sin(2kx)}{4k^2 - 1} = 1 - \frac{2x}{\pi}.
\]
After a little algebra on the left we have

\[
\frac{2}{\pi} \sum_{k=1}^{\infty} \left( \frac{4k^2 - 1}{k} \right) \sin(2kx) = \frac{4k^2 - 1}{4k^2 - 1} = 1 - \frac{2x}{\pi},
\]

which becomes

\[
\frac{2}{\pi} \sum_{k=1}^{\infty} \frac{\sin(2kx)}{k} = 1 - \frac{2x}{\pi}.
\]

Multiplying both sides by \(\pi\) we arrive at the following formula which we need to verify

\[
\frac{2}{\pi} \sum_{k=1}^{\infty} \frac{\sin(2kx)}{k} = \pi - 2x, \quad \text{for } 0 < x < \pi. \tag{5}
\]

Note the transformation

\[
s = \varphi(x) = \pi - 2x : (0, \pi) \rightarrow (-\pi, \pi)
\]

Now the hint (multiplied by 2) can be written as

\[
2 \sum_{k=1}^{\infty} \frac{(-1)^{(k+1)}}{k} \sin(k) = s, \quad -\pi < s < \pi.
\]

So we can substitute \(s = \varphi(x) = \pi - 2x\) for \(0 < x < \pi\) into the above equation and we have

\[
\begin{align*}
(\pi - 2x) &= 2 \sum_{k=1}^{\infty} \frac{(-1)^{(k+1)}}{k} \sin(k(\pi - 2x)) \\
&= 2 \sum_{k=1}^{\infty} \frac{(-1)^{(k+1)}}{k} \left( \sin(k\pi) \cos(2kx) - \sin(2kx) \cos(k\pi) \right) \\
&= 2 \sum_{k=1}^{\infty} \left[ \frac{(-1)^{(k+1)}(-1)^k}{k} \right] \frac{\sin(2kx)}{k} \\
&= 2 \sum_{k=1}^{\infty} \frac{\sin(2kx)}{k}
\end{align*}
\]

which is exactly (5) and we are done.