Method of Undetermined Coefficients

Our goal is to find the general solution of \[ ay'' + by' + cy = f(t) \] (*)

The general solution of (*) is obtained as \[ y = y_h + y_p \] where

1. \( y_h \) is the general solution of the homogeneous problem, i.e. \( y_h = c_1y_1 + c_2y_2 \) where \( y_1, y_2 \) are two linearly independent solutions of \( ay'' + by' + cy = 0 \).

2. \( y_p \) is (any) particular solution of the nonhomogeneous problem (*).

The main problem then is to find \( y_p \).

Remarks on the Method of Undetermined Coefficients

Remark:
1. The general solution of the homogeneous problem is given as a sum of numbers times terms of the form
   \[ p(t), \ p(t)e^{at}, \ p(t)e^{at}\cos(\beta t), \ p(t)e^{at}\sin(\beta t) \] (1)
   where \( p(t) \) is a polynomial in \( t \). No other types of solutions are possible!

2. This remains true for the nonhomogeneous problem (*) provided the right hand side \( f(t) \) is also given as a sum of terms of the form (1).

3. The main thing is to find \( y_p \) and here we consider the case of \( f(t) \) in the form (1).

First we consider \( f(t) \) in the form of a constant times a single term in the form (1).

\[ ay'' + by' + cy = Ct^m e^{\alpha t} \] \( \Rightarrow \) \[ y_p = t^s(A_m t^m + \cdots + A_1 t + A_0)e^{\alpha t} \]

1. \( s = 0 \) if \( r_0 \) is not a root of the characteristic polynomial \[ ar^2 + br + c = 0 \] (†).
2. \( s = 1 \) if \( r_0 \) is a simple root of (†).
3. \( s = 2 \) if \( r_0 \) is a double root of (†).

N.B. The above case includes the case \( r_0 = 0 \) in which case the right side is \( Ct^m \).

\[ ay'' + by' + cy = \begin{cases} \ C t^m e^{\alpha t} \cos(\beta t) \\ C t^m e^{\alpha t} \sin(\beta t) \end{cases} \] \( \Rightarrow \) \[ y_p = t^s(A_m t^m + \cdots + A_1 t + A_0)e^{\alpha t} \cos(\beta t) + t^s(B_m t^m + \cdots + B_1 t + B_0)e^{\alpha t} \sin(\beta t) \]

1. \( s = 0 \) if \( r_0 = \alpha \pm i\beta \) is not a root of (†).
2. \( s = 1 \) if \( r_0 = \alpha \pm i\beta \) is a simple root of (†).