A homogeneous linear differential equation with constant real coefficients of order \( n \) has the form
\[
y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0y = 0.
\]

We introduce the notation \( D = \frac{d}{dx} \) and write the above equation as
\[
P(D)y \equiv (D^n + a_{n-1}D^{n-1} + \cdots + a_0)y = 0.
\]

By the fundamental theorem of algebra we can write
\[
P(D) = \left(D - r_1\right)^{m_1} \cdots \left(D - r_k\right)^{m_k} \left(D^2 - 2\alpha_1D + \alpha_1^2 + \beta_1^2\right)^{p_1} \cdots \left(D^2 - 2\alpha_{\ell}D + \alpha_{\ell}^2 + \beta_{\ell}^2\right)^{p_{\ell}},
\]
where \( \sum_{j=1}^{k} m_j + 2 \sum_{j=1}^{\ell} p_j = n \).

**Lemma 1.** The general solution of \( (D - r)^k y = 0 \) is
\[
y = \left(c_1 + c_2 x + \cdots + c_k x^{(k-1)}\right) e^{rx}
\]
and the general solution of \( (D^2 - 2\alpha D + \alpha^2 + \beta^2)^k y = 0 \) is
\[
y = \left(c_1 + c_2 x + \cdots + c_k x^{(k-1)}\right) e^{\alpha x} \cos(\beta x) + \left(d_1 + d_2 x + \cdots + d_k x^{(k-1)}\right) e^{\alpha x} \sin(\beta x).
\]

**Proof.** Note first that \( (D - r)e^{rx} = D(e^{rx}) - re^{rx} = re^{rx} - re^{rx} = 0 \) and for \( k > j \)
\[
(D - r) \left(x^j e^{rx}\right) = D \left(x^j e^{rx}\right) - r \left(x^j e^{rx}\right) = j x^{j-1} e^{rx}.
\]
Thus we have
\[
(D - r)^k \left(x^j e^{rx}\right) = (D - r)^{k-1} \left[(D - r) \left(x^j e^{rx}\right)\right] = j(D - r)^{k-1} \left(x^{j-1} e^{rx}\right) = \cdots = j! (D - r)^{k-j} \left(e^{rx}\right) = 0 \text{ if } k > j.
\]

Therefore, each function \( x^j e^{rx} \), for \( j = 0, 1, \cdots, (k-1) \), is a solution of the equation and by the fundamental theory of algebra these functions are linearly independent, i.e.,
\[
0 = \sum_{j=1}^{k} c_j x^{j-1} e^{rx} = e^{rx} \sum_{j=1}^{k} c_j x^{j-1}, \text{ for all } x
\]
implies \( c_1 = c_2 = \cdots = c_k = 0 \).
Note that each factor \((D^2 - 2\alpha D + \alpha^2 + \beta^2)\) corresponds to a pair of complex conjugate roots \(r = \alpha \pm i\beta\). In the above calculations we did not assume that \(r\) is real so that for a pair of complex roots we must have solutions
\[
e^{(\alpha \pm i\beta)x} = e^{i\beta x}e^{\alpha x} = e^{\alpha x}(\cos(\beta x) + i\sin(\beta x)),
\]
and any linear combination of these functions will also be a solution. In particular the real and imaginary parts must be solutions since
\[
\frac{1}{2} [e^{\alpha x} (\cos(\beta x) + i\sin(\beta x))] + \frac{1}{2} [e^{\alpha x} (\cos(\beta x) - i\sin(\beta x))] = e^{\alpha x} \cos(\beta x)
\]
\[
\frac{1}{2i} [e^{\alpha x} (\cos(\beta x) + i\sin(\beta x))] - \frac{1}{2i} [e^{\alpha x} (\cos(\beta x) - i\sin(\beta x))] = e^{\alpha x} \sin(\beta x)
\]
Combining the above results we find that the functions
\[
y = (c_1 + c_2 x + \cdots + c_n x^{(n-1)}) e^{\alpha x} \cos(\beta x)
\]
and
\[
y = (d_1 + d_2 x + \cdots + d_n x^{(n-1)}) e^{\alpha x} \sin(\beta x).
\]
are solutions and these solutions are linearly independent. □

The general solution of \(P(D)y = 0\) is given as a linear combination of the solutions for each real root and each pair of complex roots.

Let us consider an example which is already written in factored form
\[
[(D + 1)^3(D^2 + 4D + 13)]y = 0
\]
The term \((D + 1)^3\) gives a part of the solution as
\[
(c_1 + c + 2x + c_3 x^2)e^{-x}
\]
and the term \((D^2 + 4D + 13)\) corresponds to complex roots with \(\alpha = -2\) and \(\beta = 3\) giving the part of the solution
\[
c_4 e^{-2x} \cos(3x) + c_5 e^{-2x} \sin(3x).
\]
The general solution is
\[
y = (c_1 + c + 2x + c_3 x^2)e^{-x} + c_4 e^{-2x} \cos(3x) + c_5 e^{-2x} \sin(3x).
\]