1. Solve the separable equation \( \frac{dy}{dx} = \frac{4y}{x^3} \).

\( \frac{dy}{y} = 4x^{-3} \, dx \quad \Rightarrow \quad \ln(y) = -2x^{-2} + C \)

2. Solve the homogeneous equation \( x^2y' = y^2 + xy + x^2 \).

\( y' = \left( \frac{y}{x} \right)^2 + \left( \frac{y}{x} \right) + 1, \quad v = \frac{y}{x} \quad \Rightarrow \quad xv' + v = v^2 + v + 1 \)

\( \frac{dv}{v^2 + 1} = \frac{dx}{x} \quad \Rightarrow \quad \tan^{-1}(v) = \ln(x) + C \quad \Rightarrow \quad \tan^{-1}\left( \frac{y}{x} \right) = \ln(x) + C \)

3. Show that \( \mu = e^{2x} \) is an integrating factor for \( 2 \cos(y) \, dx - \sin(y) \, dy = 0 \) and use it to solve (i.e., solve the resulting exact equation).

Multiple by \( e^{2x} \) to get \( 2e^{2x} \cos(y) \, dx - e^{2x} \sin(y) \, dy = 0 \). Set \( \tilde{M} = 2e^{2x} \cos(y) \) and \( \tilde{N} = -e^{2x} \sin(y) \) and note that \( \tilde{M}_y = -2e^{2x} \sin(y) = \tilde{N}_x \). So there exists an \( F \) so that

\( F_x(x, y) = 2e^{2x} \cos(y), \quad F_y(x, y) = -e^{2x} \sin(y) \)

Integrate the first by \( x \) to get \( F = e^{2x} \cos(y) + \ell(y) \). Then we have

\( -e^{2x} \sin(y) = F_y = -e^{2x} \sin(y) + \ell'(y) \quad \Rightarrow \quad \ell'(y) = 0 \quad \Rightarrow \quad \ell(y) = 0 \quad \Rightarrow \quad e^{2x} \cos(y) = C \).

4. Find an integrating factor and show that it is one (DO NOT SOLVE) \( dx + (x - e^{-y}) \, dy = 0 \).

\( q(y) = \frac{M_y - N_x}{M} = \frac{0 - 1}{1} = -1 \quad \Rightarrow \quad \mu = e^{-\int(-1) \, dy} = e^y \)

\( e^y dx + (e^y x - 1) \, dy = 0 \quad \Rightarrow \quad e^y = \tilde{M}_y = \tilde{N}_x = e^y \)

5. Solve the following First Order Linear initial value problem for \( y \)

\( y' + \frac{1}{x}y = \frac{\cos(x)}{x} \) with \( y(\pi) = 0 \).

\( \mu = e^{\int \frac{1}{x} \, dx} = e^{\ln(x)} = x \) so \( [xy]' = \cos(x) \) and \( xy = \sin(x) + C \). We evaluate the constant by

\( 0 = \sin(\pi) + C \quad \Rightarrow \quad C = 0 \) so \( xy = \sin(x) \).

6. Use the substitution \( z = (x - y) \) to solve \( y' = (x - y)^3 + 1 \).

Set \( z = x - y \quad \Rightarrow \quad z' = 1 - y' \quad \Rightarrow \quad z' = 1 - y' = 1 - [(x - y)^3 + 1] = -z^3 \) so \( -z^{-3} \, dz = dx \) and integrating both sides and substituting \( z = (x - y) \) gives

\( \frac{(x - y)^{-2}}{2} = x + C \).