1. Determine the radius of convergence of the power series $\sum_{m=1}^{\infty} \frac{4^n}{n^2} x^{2n}$

\[
\sum_{m=1}^{\infty} \frac{4^n}{n^2} x^{2n} = \sum_{m=1}^{\infty} \frac{(4x^2)^n}{n^2}
\]
so set $t = 4x^2$ to obtain $\sum_{m=1}^{\infty} \frac{t^n}{n^2}$. Thus, for $t$ we have

\[
R = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n \to \infty} \frac{(n+1)^2}{n^2} = 1
\]

so the series converges for $|4x^2| = |t| < 1$ which means for $|x| < 1/2$.

2. Consider the equation: $(x - 3)y' + 2y = 0$

(a) Find the general solution in the form $y = \sum_{n=0}^{\infty} a_n x^n$

\[
y = c_0 \sum_{n=0}^{\infty} \frac{(n+1)}{3^n x^n}.
\]

(b) Find the radius of convergence of the series.

\[
y = \sum_{n=0}^{\infty} (n+1) \left( \frac{x}{3} \right)^n t = \sum_{n=0}^{\infty} (n+1)t^n
\]
so

\[
R = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n \to \infty} \frac{n+1}{n+2} = 1
\]
and the series converges for $|t| < 1$ or $|x| < 3$.

(c) Sum the series to obtain a “closed form solution.” (hint: use a geometric series formula).

\[
\sum_{n=0}^{\infty} (n+1)t^n = \frac{d}{dt} \left( \sum_{n=0}^{\infty} t^{n+1} \right) = \frac{d}{dt} \left( t \sum_{n=0}^{\infty} t^n \right) = \frac{d}{dt} \left( \frac{t}{1-t} \right) = \frac{1}{(1-t)^2},
\]
So we get $y = \frac{9c_0}{(3 - x)^2}$. 
3. Find a recursion formula for a solution of \((x^2 - 1)y'' + 2xy' + 2xy = 0\) in the form \(y = \sum_{m=0}^{\infty} a_m x^m\).

Then find the first three nonzero terms in each of two linearly independent solutions.

**Answer:**

\[ c_2 = c_3 = 0, \quad c_{n+4} = \frac{(n+1)c_{n+1} - c_n}{(n+3)(n+4)} \text{ for } n \geq 0. \]

\[ y_1 = 1 - \frac{1}{12} x^4 + \frac{1}{126} x^7 + \cdots, \quad y_2 = x - \frac{1}{12} x^4 - \frac{1}{20} x^5 + \cdots. \]

4. Use the Taylor method to obtain the first three nonzero terms in a series solution for the equation \(y' = \sin(y) + e^x, \quad y(0) = 0\).

**Answer:**

We seek \(y = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{y^{(n)}(0)}{n!} x^n\). Then we have \(y(0) = 0\) implies that \(a_0 = 0\). Then \(y' = \sin(y) + e^x\) implies \(y'(0) = \sin(0) + e^0 = 1\) so \(a_1 = 1\). Next, we differentiate the equation to obtain \(y'' = y' \cos(y) + e^x\) so that \(y''(0) = 1 + 1 = 2\) and \(a_2 = \frac{2}{2} = 1\). Finally, we differentiate again to get \(y''' = y'' \cos(y) - (y')^2 \sin(y) + e^x\) so that \(y'''(0) = 2 + 1 = 3\) and \(a_3 = \frac{3}{3!} = \frac{1}{2}\). Thus we have

\[ y = x + x^2 + \left(\frac{1}{2}\right) x^3 + \cdots. \]

5. Investigate the nature of the possible singular point at \(x = 0\) for the equations. Also find the indicial equation and describe the possible form of the solutions.

(a) \(x^2 y'' + \cos(x)y' + xy = 0\)

**Answer:** \(x = 0\) is an irregular singular point. Nothing we can say.

(b) \(x^2(x + 2)y'' - xy' + (1 + x)y = 0\)

**Answer:** The indicial equation reduces to \(2r^2 - 3r + 1 = 0\) so we get \(r_1 = 1\) and \(r_2 = 1/2\). Therefore \(y_1 = x \sum_{n=0}^{\infty} a_n x^n\) (with \(a_0 \neq 0\)), and

\[ y_2 = x^{1/2} \sum_{n=0}^{\infty} A_n x^n \text{ (with } A_0 \neq 0) \]
(c) \(x(1 + x)y'' + 2y' + 3xy = 0\)

The indicial equation reduces to \(r^2 + r = 0\) so we get \(r_1 = 0\) and \(r_2 = -1\). Since these roots differ by an integer so we have one solution (note \(r_2 = 0\) is the larger root) \(y_1(x) = \sum_{n=0}^{\infty} a_n x^n\) (with \(a_0 \neq 0\)), and

\[ y_2(x) = k y_1(x) \ln(x) + x^{-1} \sum_{n=0}^{\infty} A_n x^n \] (with \(A_0 \neq 0\)).

6. Find the Frobenius series solutions of

(a) \(4xy'' + 2y' + y = 0\)

\[ y_1 = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n)!} = \cos(\sqrt{x}), \]

\[ y_2 = x^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n + 1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{(2n+1)/2}}{(2n+1)!} = \sin(\sqrt{x}). \]

(b) \(4xy'' + 8y' + xy = 0\)

The indicial equation is \(r^2 + r = 0\) so \(r_1 = 0\) and \(r_2 = -1\). Thus we are in case 3. After lots of work we find:

\[ y_1 = \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n}}{2(2n + 1)!} = \frac{\sin(x/2)}{x}, \]

\[ y_2 = x^{-1} \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n}}{(2n)!} = \frac{\cos(x/2)}{x}. \]