January 12, 2001 Homework Perturbation Theory # 1
Math 5311 Sec. 1,

1. Find the first three terms in the asymptotic expansion for the roots of \(x^3 - 5x^2 + 4x + \epsilon = 0\). Note that \(x^3 - 5x^2 + 4x = 0\) has roots \(x = 0, x = 1\) and \(x = 4\).

2. Use the quadratic formula to obtain the roots of \(x^2 - 2\epsilon x - \epsilon = 0\) (see Example 5.3) and use it to verify our findings in the notes. Namely that the two perturbed roots have have expansions

\[x_1(\epsilon) = \epsilon^{1/2} + \epsilon + \frac{1}{2} \epsilon^{3/2} + \mathcal{O}(\epsilon^2),\]
\[x_2(\epsilon) = -\epsilon^{1/2} + \epsilon - \frac{1}{2} \epsilon^{3/2} + \mathcal{O}(\epsilon^2).\]

3. Find the first two terms in the expansion of the roots of \(x^3 - \epsilon x^2 - \epsilon^2 = 0\).

4. Show that the equation (which is in the form (5.2.3) from the notes)

\[P(x, \epsilon) = 1 + \epsilon^{-1}x + \epsilon^{-1}x^2 + x^3 = 0\]

has roots that approach zero, a finite number and infinity.

5. For small \(\epsilon\), find the first two terms in the expansion of each of the roots and compare with the approximate answers obtained from Maple with \(\epsilon = .01\)

\[P(x, \epsilon) = x^3 - (3 + \epsilon)x - 2 + \epsilon = 0\]

6. For small \(\epsilon\), find the first two terms in the expansion of each of the roots and compare with the approximate answers obtained from Maple with \(\epsilon = .01\)

\[P(x, \epsilon) = \epsilon x^4 - x^3 + 3x - 2 = 0.\]

7. Sketch the graph of \(f_\epsilon(x) = x^2 + e^{\epsilon x}\) for \(x \in \mathbb{R}\) and a small positive \(\epsilon\). Use a little calculus to convince yourself that the graph is correct. From the graph note that the equation

\[x^2 + e^{\epsilon x} = 5\]

has two roots for small positive \(\epsilon\) and find the first two terms in an asymptotic expansion of these roots.

8. Determine a two term expansion for the large roots of \(x \tan(x) = 1\)