4 Second Order Boundary Value Problems

4.1 Oscillation and Separation Theory

Consider the differential equation

\[ a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \]  \hspace{1cm} (4.1.1)

where \( a_2(x) \) is not zero for all \( x \in [a, b] \), \( a_i(x) \in C[a, b] \).

We consider this equation in the Hilbert space \( L^2(a, b) \) with inner product

\[ \langle f, g \rangle = \int_a^b f(x)\overline{g(x)} \, dx. \]

If we define the operator

\[ M(y) = a_2y'' + a_1y' + a_0y \]  \hspace{1cm} (4.1.2)

then the formal adjoint of \( M \) is defined by

\[ \overline{M}(y) = (a_2y)'' - (a_1y)' + a_0y \]

which can be expanded to give

\[ \overline{M}(y) = a_2y'' + (2a_2' - a_1)y' + (a_2'' - a_1' + a_0)y. \]

After some manipulation it can be shown that the expression is an exact derivative. That is we have that

\[ vM(u) - u\overline{M}(v) = [(a_1 - a_2')vu + a_2(vu' - uv')]'. \]

This result is refered to as the Lagrange’s identity. We rewrite it as

\[ vM(u) - u\overline{M}(v) = \frac{d}{dx}P(u, v) \]

and integrate from \( a \) to \( b \) to obtain Green’s formula,

\[ \int_a^b [vM(u) - u\overline{M}(v)] \, dx = P(u, v)|_a^b \]

where on the right side the notation mean

\[ P(u, v)|_a^b = P(u(b), v(b)) - P(u(a), v(a)). \]
If $M(u) = M(v)$, the operator $M$ is said to be self-adjoint and the equation $M(u) = 0$ is also called self-adjoint.

Note that if $a'_2 = a_1$, then $M$ is selfadjoint and in this case Lagrange’s identity becomes

$$vM(u) - uM(v) = [a_2 (vu' - uv')]' = \frac{d}{dx} [a_2(x)W(v, u)]$$

(4.1.3)

(where $W$ is the Wronskian of $u$ and $v$) and

$$M(u) = (a_2 u')' + a_0 u.$$  

Next we show that every general linear second order equation (4.1.1) with $a_2 \neq 0$ can be put into self-adjoint form.

Rewrite (4.1.1) in the form

$$y'' + \frac{a_1}{a_2} y' + \frac{a_0}{a_2} y = 0$$

or, setting

$$P(x) = \frac{a_1}{a_2}, \quad Q(x) = \frac{a_0}{a_2}$$

we have

$$y'' + P(x)y' + Q(x)y = 0.$$  

(4.1.4)

Define

$$p(x) = e^{\int P(s) ds},$$

Then

$$\frac{d}{dx} (p(x)y') + p(x)Q(x)y = 0$$

or

$$(py')' + q(x)y = 0$$  

(4.1.5)

where $q = pQ$. Clearly the operator $L$ defined by

$$L(y) = (py')' + q(x)y$$

is self-adjoint.

This idea can be pushed one step further to show that any equation of the form (4.1.1) can be written in the form

$$Lu = u'' + q(x)u = 0.$$  

(4.1.6)

The form (4.1.1) is usually refered to as the standard form and (4.1.6) is called the normal form.

To see how to obtain (4.1.6) we proceed as follows: Let $y = uv$ so that

$$y' = u'v + uv'.$$
and
\[ y'' = uv'' + 2u'v' + u''v. \]
Substituting these expressions into (4.1.4) we have
\[ vu'' + (2v' + Pv)u' + (v'' + Pv' + Qv)u = 0. \] (4.1.7)

Setting the coefficient of \( u' \) equal to zero we have
\[ v' + \frac{1}{2}Pv = 0 \]
which implies
\[ v = \exp \left( -\frac{1}{2} \int P(x) \, dx \right). \]
Substituting this value for \( v \) in (4.1.7) and then dividing both sides by \( v \) we obtain (4.1.6) with
\[ q(x) = Q - \frac{1}{2}P(x)^2 - \frac{1}{4}P'(x). \]

Note that \( v \) is never zero so that this makes sense and also since it is positive it has no effect on the zeros of the solutions \( y \) of (4.1.4). So it is equivalent to study the properties of \( u \).

4.2 Separation Theorems

**Theorem 4.1.** [Sturm Separation Theorem] Consider the equation (4.1.6).

1. A nontrivial solution of (4.1.6) can have at most a finite number of zeros on \([a,b]\).
2. All zeros of a solution are simple.
3. If \( u_1(x), u_2(x) \) are linearly independent solutions of (4.1.6) then between any two zeros of \( u_1(x) \) there is precisely one zero of \( u_2(x) \).

**Proof.** 1. Suppose there exists infinitely many zeros, \( \{z_n\} \), since \([a,b]\) is a bounded interval we can select a subsequence \( \{x_n\} \) such that \( x_n \to x_0 \). Then
\[ 0 = \lim_{n \to \infty} y(x_n) = y(x_0). \]
Also,
\[ y'(x_0) = \lim_{n \to \infty} \frac{y(x_n) - y(x_0)}{x_n - x_0} = 0 \]
and so \( y \) is a solution of \( Ly = 0, y(x_0) = 0 \) and \( y'(x_0) = 0 \). But \( y \equiv 0 \) is also a solution of this IVP, so \( y(x) = 0 \) by the Fundamental Existence Uniqueness Theorem.

2. Suppose that \( y \) is a solution with a double root at a point \( x_0 \in [a,b] \). This means that \( y(x_0) = 0 \) and \( y'(x_0) = 0 \) (where this is a one sided derivative if \( x_0 = a \) or \( b \)). Now just as in part 1) we can appeal to the Fundamental Existence Uniqueness Theorem to conclude that \( y(x) = 0 \).
3. Suppose $x_0$, $x_1$ are consecutive zeros of $u_1(x)$, and assume that $x_0 < x_1$. Then $u_2(x_0) \neq 0$ and $u_2(x_1) \neq 0$, or else

$$W(u_1, u_2)(x_i) = \begin{vmatrix} u_1(x_i) & u_2(x_i) \\ u_1'(x_i) & u_2'(x_i) \end{vmatrix} = 0, \quad i = 1, 2.$$ 

So, without loss of generality assume $u_1(x) > 0$ for $x \in (x_0, x_1)$, and $u_2(x) > 0$. We must have $W(x_0) < 0$ since $u_1'(x_0) > 0$ and $u_2(x_0) > 0$ and

$$W(u_1, u_2)(x_0) = -u_1'(x_0)u_2(x_0) < 0.$$ 

Now the Wronskian of $u_1, u_2$ cannot change sign, since otherwise it would have to have a zero at some point which contradicts the fact that $u_1$ and $u_2$ are linearly independent, so we must have also have $W(x_1) < 0$.

But

$$W(u_1, u_2)(x_1) = -u_1'(x_1)u_2(x_1),$$

and since $u_1'(x_1) < 0$ this implies that $u_2(x_1) < 0$. Hence $u_2(x)$ must vanish in $(x_0, x_1)$.

If we apply the argument with the roles of $u_1$ and $u_2$ interchanged, we see that between two consecutive zeros of $u_2$ there must be a zero of $u_1(x)$. Hence the zeros of $u_1$ and $u_2$ must interlace.

\[\square\]

**Remark 4.1.** 1. Roughly speaking, the Sturm Separation theorem states that linearly independent solutions of a single equation have the same number of zeros.

2. On the other hand, if we consider two different equations, for example

$$y'' + y = 0, \quad y'' + 4y = 0$$

then solutions of the second equation oscillate more rapidly than those of the first.

### 4.3 Oscillation Theory

More generally, Sturm Comparison theorems address the rate of oscillation of solutions of different equations.

As mentioned above we shall consider equations of the form (4.1.6)

$$L(y) = y'' + qy = 0, \quad a < x < b$$

where $q \in C^0[a, b]$.

**Theorem 4.2.** Let $\begin{cases} L_1(u_1) = u_1'' + q_1u_1 = 0, \\ L_2(u_2) = u_2'' + q_2u_2 = 0 \end{cases}$ where $q_2(x) \geq q_1(x)$, for all $x \in [a, b]$. Then between any two consecutive zeros $x_1, x_2$ of $u_1(x)$ there is a zero of $u_2(x)$ unless $q_1(x) = q_2(x)$ on $[x_1, x_2]$ and, in this case, $u_2$ and $u_2$ are linearly dependent on $[x_1, x_2]$. 

Remark 4.2. This tells us that either \( u_2 \) has a zero between \( x_1 \) and \( x_2 \) or \( u_2(x_1) = u_2(x_2) = 0 \).

Proof. Suppose \( u_1(x_1) = u_1(x_2) = 0 \) (where \( x_1 \) and \( x_2 \) are consecutive zeros of \( u_1 \)) and \( u_2(x) \neq 0 \) on \( (x_1, x_2) \).

Let us now assume that \( u_2(x) \neq 0 \) on \( (x_1, x_2) \). Our goal is to show that under this assumption \( q_1 = q_2 \) and \( u_1 \) and \( u_2 \) are linearly dependent.

Without loss of generality assume that \( u_1, u_2 > 0 \) on \( (x_1, x_2) \). Note that this implies

\[
\begin{align*}
\frac{d}{dx}u_1(x_1) > 0 & \quad \text{and} \quad \frac{d}{dx}u_1(x_2) < 0
\end{align*}
\]

as depicted in the following figure.

Let us denote by \( W(x) \) the Wronskian of \( u_1 \) and \( u_2 \) evaluated at \( x \), i.e.,

\[
W(x) = u_1(x)u'_2(x) - u_2(x)u'_1(x).
\]

Under the assumption that \( q_2 \geq q_1 \) we have

\[
\begin{align*}
\frac{d}{dx}W(x) &= \frac{d}{dx}(u_2u'_1 - u_1u'_2) = u_2u''_1 + u_2u''_1 - u'_2u'_1 - u_1u''_2 \\
&= u_2u''_1 - u_1u''_2 = u_2(-q_1u_1) - (-q_2u_2)u_1 \\
&= u_1u_2(q_1 - q_1) \geq 0
\end{align*}
\]

We see that \( W \) is nondecreasing on \( (x_1, x_2) \). However, as we have already mentioned (see the figure), since \( u_1(x_1) = u_1(x_2) = 0 \) and \( u_1 > 0 \) on \( (x_1, x_2) \), \( u'_1(x_1) > 0 \), \( u'_1(x_2) < 0 \). Finally since \( u_2(x) > 0 \) on \( (x_1, x_2) \), we have

\[
W(x_1) = u_2(x_1)u'_1(x_1) \geq 0,
\]

and

\[
W(x_2) = u_2(x_2)u'_1(x_2) \leq 0.
\]

Now since \( W \) is nondecreasing, the only way it can be greater than zero at \( x_1 \) and less than or equal to zero at \( x_2 \) is for it to be identically zero. Thus we have

\[
W(x) \equiv 0, \quad x \in [x_1, x_2].
\]
From (4.3.1) we conclude that

\[ q_2 - q_1 = 0 \]

so that \( q_1 = q_2 \). Therefore the two equations are the same and we conclude that \( u_1 \) and \( u_2 \) satisfy the same linear ordinary differential equation whose Wronskian vanishes identically. Therefore from the corollary to Abel’s formula in the theory of ODEs we have that \( u_1 \) and \( u_2 \) are linearly dependent.

A more general version of this theorem is

**Theorem 4.3.** Assume \( p, q \in C^0[a, b] \) and \( z(x) \) is a non trivial solution of

\[ z'' + q(x)z = 0 \]

where

\[ z(a) = z(b) = 0. \]

If

\[ \int_a^b (p - q)z^2 dx \geq 0 \]

then a nontrivial solution of

\[ y'' + p(x)y = 0 \]

\[ y(a) = 0 \]

has a zero in the interval \((a, b]\).

**Proof.** Suppose \( y(x) \neq 0 \) in \((a, b]\). Then

\[ z(z'' + qz) = 0 \]

\[ \frac{z^2}{y}(y'' + py) = 0 \]

or

\[ zz'' - z^2 \frac{y''}{y} = z^2(p - q) \]

or

\[ \frac{z}{y}(yz' - zy')' = z^2(p - q). \]

Now note that

\[ \lim_{x \to a} \frac{z(x)}{y(x)} = \lim_{z \to a} \frac{z'(a)}{y'(a)} \]

and since \( z'(a) \neq 0, y'(a) \neq 0, z', y' \in C[a, b] \), this limit exists and is finite. Hence it makes sense to write

\[ \int_a^b \frac{z}{y}(yz' - zy')' dx = \int_a^b z^2(p - q) dx \geq 0. \]
Now integrate by parts and since \( z(b) = 0, y(b) \neq 0, z(a) = y(a) = 0, \)
\[
\frac{z}{y}(yz' - zy')|_a^b - \int_a^b (yz' - zy')(\frac{z}{y})' = \int_a^b z^2(p - q)dx \geq 0, 
\]
\([*)\]
\[
\text{or} \quad -\int_a^b \frac{(yz' - zy')^2}{y^2}dx \geq 0
\]
\[
\text{or} \quad 0 \geq \int_a^b \frac{(yz' - zy')^2}{y^2}dx.
\]
The right hand side is identically zero if \( y(x) = cz(x) \) in which case the result is trivially true. So if \( y(x) \neq cz(x) \), the right hand side is positive and we get a contradiction. Hence \( y(x) \) must vanish in \((a, b]\).

The proof shows that if \( p(x) \neq q(x) \) then
\[
\int_a^b z^2(p - q)dx > 0.
\]
In this case \( y(x) \) must have a zero in \((a, b]\). If not, then just as before we could derive \([*)\] by dividing by \( y(x) \) and the boundary term in \([*)\] would vanish since \( y(b) = 0 \), and we would obtain
\[
\int_a^b \frac{(yz' - zy')^2}{y^2}dx < 0,
\]
which is a contradiction.

### 4.4 Sturm-Liouville Boundary Value Problems

In practice one often encounters a second order differential equation in so-called self-adjoint form and generally one finds that the most common boundary conditions are either separated or periodic. As we have already seen, a second order operator \( L \) is in self-adjoint form if
\[
L(y) = (py')' + r(x)y.
\]
We are particularly interested in BVP’s of the form
\[
L(y) + \lambda g(x)y = 0, \quad a < x < b, 
\]
\[
B_1(y) = 0, \quad B_2(y) = 0,
\]
where \( p, p', g, r \) are real and continuous on \([a, b]\), and \( p > 0 \) on \([a, b]\). The corresponding separated boundary conditions are given by
\[
B_1(y) = \alpha_1 y(a) + \alpha_2 y'(a), \quad (4.4.3)
\]
\[
B_2(y) = \beta_1 y(b) + \beta_2 y'(b). \quad (4.4.4)
\]
The BVP (4.4.1)-(4.4.4) is called a Regular Sturm-Liouville Eigenvalue Problem. The values of \( \lambda \) for which the BVP has a nontrivial solution are called eigenvalues.

It is useful to consider some other forms of (4.4.1) encountered in the literature. The first reduction is related to our earlier discussion in which we showed that the problem can be replaced with one for which \( p = 1 \). For this discussion we start with the equation

\[
\frac{d}{dz} \left( \tilde{p}(z) \frac{dy}{dz} \right) + (\tilde{r}(z) + \lambda \tilde{g}(z)) y = 0.
\]  

(4.4.5)

To obtain the first canonical form, we define new independent variable by

\[
\zeta = \int_a^z \frac{dt}{\tilde{p}(t)},
\]  

(4.4.6)

Note that, by the chain rule,

\[
\frac{d}{dz} \left( \tilde{p} \frac{dy}{dz} \right) = \frac{d}{dz} \left( \tilde{p} \frac{dy}{d\zeta} \frac{d\zeta}{dz} \right) = \frac{d}{dz} \left( \frac{dy}{d\zeta} \right) = \frac{1}{\tilde{p}} \frac{d^2 y}{d\zeta^2},
\]

and the equation (4.4.5) is transformed into

\[
\frac{d^2 y}{d\zeta^2} + (\tilde{p}(z)\tilde{r}(z) + \lambda \tilde{p}(z)\tilde{g}(z)) y = 0.
\]  

(4.4.7)

Our goal is to write this equation in the form

\[
\frac{d^2 u}{d\xi^2} + (q(\xi) + \lambda) u = 0.
\]  

(4.4.8)

To this end we introduce new dependent and independent variables by

\[
y = k(\zeta) u(\xi), \quad \xi = \int_0^\zeta \frac{dt}{k^2(t)},
\]  

(4.4.9)

where \( k \) will be selected below.

Thus we can write

\[
\frac{dy}{d\zeta} = u \frac{dk}{d\zeta} + k \frac{du}{d\zeta} = u \frac{dk}{d\zeta} + k \frac{du}{d\zeta} \frac{d\zeta}{d\xi} = u \frac{dk}{d\zeta} + \frac{1}{k} \frac{du}{d\xi}.
\]

Differentiating again we have

\[
\frac{d^2 y}{d\zeta^2} = \left( u \frac{d^2 k}{d\zeta^2} + \frac{dk}{d\zeta} \left( \frac{du}{d\zeta} \frac{d\zeta}{d\xi} \right) \right) + \left( - \frac{1}{k^2} \frac{dk}{d\zeta} \frac{du}{d\zeta} + \frac{1}{k} \frac{d}{d\zeta} \left( \frac{du}{d\xi} \right) \right)
\]

\[
= \left( u \frac{d^2 k}{d\zeta^2} + \frac{1}{k^2} \frac{du}{d\zeta} \frac{dk}{d\xi} \right) + \left( - \frac{1}{k^2} \frac{dk}{d\zeta} \frac{du}{d\xi} + \frac{1}{k} \frac{1}{k^2} \frac{d^2 u}{d\xi^2} \right)
\]

\[
= u \frac{d^2 k}{d\zeta^2} + \frac{1}{k^3} \frac{d^2 u}{d\xi^2}.
\]
So, from (4.4.7), we have

\[
0 = \frac{d^2 y}{d\zeta^2} + (\tilde{p}(z)\tilde{r}(z) + \lambda\tilde{p}(z)\tilde{g}(z))y
\]

\[
= \left( u \frac{d^2 k}{d\zeta^2} + \frac{1}{k^3} \frac{d^2 u}{d\zeta^2} \right) + (\tilde{p}(z)\tilde{r}(z) + \lambda\tilde{p}(z)\tilde{g}(z))ku.
\]

This implies

\[
\frac{1}{k^3} \frac{d^2 u}{d\zeta^2} + \left( \frac{1}{k} \frac{d^2 k}{d\zeta^2} + \tilde{p} + \lambda\tilde{g} \right) ku = 0.
\]

At this point we choose \( k \) by

\[
k^4\tilde{p}\tilde{g} = 1, \quad \text{i.e.,} \quad k(\zeta) = \frac{1}{(\tilde{p}(z)\tilde{g}(z))^{1/4}}, \quad (4.4.10)
\]

let

\[
q(\xi) = k^4(\zeta) \left( \tilde{p}(z)\tilde{r}(z) + \frac{1}{k} \frac{d^2 k}{d\zeta^2} \right) \quad (4.4.11)
\]

and we obtain (4.4.8).

**Example 4.1.** Consider the equation

\[
y'' + \frac{1}{z} y' + \left( \lambda - \frac{N^2}{z^2} \right)y = 0
\]

so that \( P = 1/z \) and \( Q = \lambda - N^2/z^2 \).

1. Our first step is to get rid of the \( y' \) term. Let us define

\[
\tilde{p} = \exp \left( \int \frac{1}{z} \ dz \right) = e^{\ln z} = z.
\]

Thus we obtain

\[
\frac{d}{dz} \left( \tilde{p} \frac{dy}{dz} \right) + (\tilde{r} + \tilde{g}\lambda) y \equiv \frac{d}{dz} \left( z \frac{dy}{dz} \right) + \left( z\lambda - \frac{N^2}{z} \right)y = 0.
\]

2. Now from (4.4.6) we let

\[
\zeta = \int_a^z \frac{1}{\tilde{p}(t)} \ dt = \int_a^z \frac{1}{t} \ dt = \ln(z) - \ln(a),
\]

which implies \( z = ae^\zeta \) and \( \zeta \) varies from \( 0 \) to \( \ln \left( \frac{b}{a} \right) \).

So in terms of our earlier notation we have

\[
\tilde{p}\tilde{g} = z^2, \quad \tilde{p}\tilde{r} = -\frac{zN^2}{z} = -N^2,
\]

and we obtain

\[
\frac{d^2 y}{d\zeta^2} + (\lambda a^2 e^{2\zeta} - N^2)y = 0.
\]
3. Finally, in the last step, we find $k$ from

$$k^4 \tilde{p} g = 1 \implies k = \left( \frac{e^{-2\zeta}}{a^2} \right)^{1/4} = \frac{1}{\sqrt{a}} e^{-\zeta/2}$$

and we seek

$$y = k(\zeta)u(\xi) = \frac{1}{\sqrt{a}} e^{-\zeta/2} u(\xi), \quad \xi = \int_0^\zeta \frac{1}{k^2(t)} \, dt = a \int_0^\zeta e^t \, dt = a (e^\zeta - 1),$$

so that $\zeta$ varies from 0 to $(b - a)$. Now

$$\frac{1}{k} \frac{d^2 k}{d\zeta^2} = \frac{1}{k} \frac{1}{4 \sqrt{a}} e^{-\zeta/2} = \frac{1}{4 k} = \frac{1}{4},$$

and

$$q(\xi) = k^4 \left( \tilde{p} r + \frac{1}{k} \frac{d^2 k}{d\zeta^2} \right) = \frac{1}{a^2} e^{-2\zeta} \left( -N^2 + \frac{1}{4} \right) = \frac{(1/4 - N^2)}{(\xi + a)^2}.$$

Finally we arrive at

$$\frac{d^2 u}{d\xi^2} + \left( \frac{(1/4 - N^2)}{\xi^2} + \lambda \right) u = 0.$$

**Definition 4.1.** We now denote by $L$ the operator

$$L = \frac{d^2}{dx^2} + q(x) \quad (4.4.12)$$

on a finite interval $[a, b]$ and define the **Regular Sturm-Liouville Boundary Value Problem** to be the problem of finding pairs $(\lambda, \varphi)$ with $\varphi \neq 0$, called an eigenpair, satisfying

$$L\varphi + \lambda \varphi = 0 \quad (4.4.13)$$

and the boundary conditions

$$B_1(\varphi) = \alpha_1 \varphi(a) + \alpha_2 \varphi'(a) = 0, \quad B_2(\varphi) = \beta_1 \varphi(b) + \beta_2 \varphi'(b) = 0, \quad (4.4.14)$$

with $\alpha_1^2 + \alpha_2^2 \neq 0$ and $\beta_1^2 + \beta_2^2 \neq 0$.

The equation together with the periodic boundary conditions

$$\varphi(a) = \varphi(b), \quad \varphi'(a) = \varphi'(b) \quad (4.4.15)$$

is called a **Periodic Sturm-Liouville Boundary Value Problem**.

In each case $\lambda$ is called an eigenvalue and $\varphi$ the eigenfunction.

**Lemma 4.1.** For $L = \frac{d^2}{dx^2} + q(x)$, on $L^2(a, b)$ with inner product

$$\langle f, g \rangle = \int_a^b f(x) \overline{g(x)} \, dx$$

and boundary conditions (4.4.14). Assume that $q$ is a real valued function defined on $[a, b]$

The for all $\varphi, \psi \in C^2(a, b)$ that satisfy the boundary conditions

$$\langle L\varphi, \psi \rangle = \langle \varphi, L\psi \rangle. \quad (4.4.16)$$

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Proof. Our main tool will be integration by parts. We write

\[ \langle \varphi, \psi \rangle = \int_a^b \varphi''(x)\overline{\psi(x)} \, dx + \int_a^b q(x)\varphi(x)\overline{\psi(x)} \, dx \]

\[ = \varphi'(x)\overline{\psi(x)} \bigg|_a^b - \int_a^b \varphi'(x)\overline{\psi'(x)} \, dx + \int_a^b q(x)\varphi(x)\overline{\psi(x)} \, dx \]

\[ = \varphi'(x)\overline{\psi(x)} \bigg|_a^b - \varphi(x)\overline{\psi'(x)} \bigg|_a^b + \int_a^b \varphi(x)\overline{\psi''(x)} \, dx + \int_a^b q(x)\varphi(x)\overline{\psi(x)} \, dx \]

\[ = \left[ \varphi'(x)\overline{\psi(x)} - \varphi(x)\overline{\psi'(x)} \right] \bigg|_a^b + \langle \varphi, L\psi \rangle. \]

We show that if the functions satisfy the boundary conditions then boundary terms are zero. In our analysis we consider only the boundary conditions at \( x = a \) (the same analysis applies at \( x = b \)) and we consider two cases: \( \alpha_2 = 0 \) and \( \alpha_2 \neq 0 \).

1. Assume that \( \alpha_2 = 0 \), then we must have \( \alpha_1 \neq 0 \) and the boundary condition at \( x = a \) becomes \( \varphi(a) = \psi(a) = 0 \). In this case we have

\[ \varphi'(a)\psi(a) - \varphi(a)\psi'(a) = 0. \]

2. Assume that \( \alpha_2 \neq 0 \), then we can write the boundary condition as

\[ \varphi'(a) = -\frac{\alpha_1}{\alpha_2} \varphi(a), \quad \psi'(a) = -\frac{\alpha_1}{\alpha_2} \psi(a), \]

and therefore

\[ \left[ \varphi'(a)\overline{\psi(a)} - \varphi(a)\overline{\psi'(a)} \right] = -\frac{\alpha_1}{\alpha_2} \left[ \varphi(a)\overline{\psi(a)} - \varphi(a)\overline{\psi(a)} \right] = 0. \]

\[ \square \]

Theorem 4.4. The eigenvalues of the Regular Sturm-Liouville Boundary Value Problem are real.

Proof. If \((\lambda, u)\) is an eigenpair then \( u \neq 0 \) and

\[ Lu + \lambda u = 0, \]

the

\[ \lambda \langle u, u \rangle = \langle \lambda u, u \rangle = \langle Lu, u \rangle = \langle u, Lu \rangle = \langle u, \lambda u \rangle = \overline{\lambda} \langle u, u \rangle. \]

Thus we have

\[ (\lambda - \overline{\lambda})\|u\|^2 = 0 \quad \Rightarrow \quad \lambda = \overline{\lambda}. \]

\[ \square \]
**Theorem 4.5.** Let \((\lambda, u), (\mu, v)\) denote a pair of eigenvalues and eigenfunctions of the Regular Sturm-Liouville Boundary Value Problem with \(\lambda \neq \mu\),

\[L(y) + \lambda y = 0\]

satisfying the boundary conditions (4.4.14). Then (from Lemma 4.1 we know that the eigenvalues are real) we have

\[(\lambda - \mu) \int_a^b u(x)v(x)dx = 0,\]

i.e., \(u\) and \(v\) are orthogonal.

**Proof.** From Lemma 4.1, we know that

\[\lambda \langle u, v \rangle = \langle \lambda u, v \rangle = \langle u, \lambda v \rangle = \langle u, v \rangle = \mu \langle u, v \rangle\]

So we conclude that \((\lambda - \mu)\langle u, v \rangle = 0\) but \(\mu = \mu\) from Lemma 4.1.

**Definition 4.2.** An eigenvalue \(\lambda\) is said to be simple if the dimension of the null space of \(L + \lambda I\) is one, i.e., \(\dim \{ \varphi : (L + \lambda I)\varphi = 0 \}\) is one. Otherwise \(\lambda\) is a multiple eigenvalue.

**Theorem 4.6.** The eigenvalues of the Regular Sturm-Liouville Boundary Value Problem are simple.

**Proof.** Suppose \(\lambda\) is an eigenvalue which is not simple and \(u, v\) are eigenfunctions corresponding to \(\lambda\). Then due to the boundary condition at \(x = a\), we have

\[
\begin{pmatrix}
u(a)
\end{pmatrix}
\begin{pmatrix}
u'(a)
\end{pmatrix}
\begin{pmatrix}
\alpha_1 \\
\alpha_2
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0
\end{pmatrix}
\]

with \(\alpha_1^2 + \alpha_2^2 \neq 0\), where \(u, v\) satisfy

\[L(y) + \lambda y = 0.\]

This implies that the determinant of the coefficient matrix must be zero and, by the corollary to Abel’s Theorem, hence \(u, v\) must be linearly dependent, i.e., \(v(x) = cu(x)\).

If, instead of the separated boundary conditions, we consider the periodic boundary conditions,

\[y(a) = y(b), \quad y'(a) = y'(b),\]

then Theorems 4.5-4.4 are still true but Theorem 4.6 is no longer true.

We have yet to address whether the Regular Sturm-Liouville Boundary Value Problem has any eigenvalues. The following examples indicate that there are infinite, but a countable number, of eigenvalues.
Example 4.2. Find the eigenvalues and eigenfunctions for

\[ y'' + \lambda y = 0, \quad 0 < x < \ell \]

with the boundary conditions

1. \( y(0) = 0, \quad y(\ell) = 0. \)

   (a) If \( \lambda = 0, \) \( y(x) = ax + b \) and the boundary conditions imply \( a = b = 0. \)

   (b) If \( \lambda < 0, \) say \( \lambda = -\mu^2, \) then

   \[ y(x) = a \sinh(\mu x) + b \cosh(\mu x). \]

   Then \( y(0) = 0 \) only if \( b = 0 \) and \( y(\ell) = 0 \) only if \( a = 0. \)

   (c) If \( \lambda > 0, \) say \( \lambda = \mu^2, \) then

   \[ y(x) = a \sin(\mu x) + b \cos(\mu x). \]

   and \( y(0) = 0 \) implies \( b = 0. \)

   To satisfy the boundary condition at \( x = \ell \) we need

   \[ a \sin(\mu \ell) = 0. \]

   We get a nontrivial solution, in this case, if

   \[ \mu \ell = n\pi, \quad n = \pm 1, \pm 2, \pm 3, \cdots \]

   Hence the eigenvalues and eigenfunctions are given by

   \[ \lambda_n = (n\pi/\ell)^2, \quad y_n(x) = \sin(n\pi x/\ell), \quad n = 1, 2, 3, \cdots. \]

2. \( y(0) = 0, \quad y'(\ell) = 0 \)

   As above it is easy to verify that eigenvalues must be positive. If \( \lambda = \mu^2 > 0, \) then

   \[ y(x) = a \sin(\mu x) + b \cos(\mu x). \]

   and \( y(0) = 0 \) implies \( b = 0. \) The second boundary condition gives

   \[ a\mu \cos(\mu \ell) = 0 \]

or

\[ \mu \ell = \left( n + \frac{1}{2} \right) \pi, \quad n = 0, \pm 1, \pm 2, \cdots \]

or

\[ \lambda_n = \left( \left( n + \frac{1}{2} \right) \frac{\pi}{\ell} \right)^2, \quad n = 0, \pm 1, \pm 2, \cdots \]

and

\[ y_n(x) = \sin \left( \left( n + \frac{1}{2} \right) \frac{\pi}{\ell} x \right). \]
3. \( y'(0) = y'(|\ell|) = 0 \)

(a) If \( \lambda = 0 \), then \( y = ax + b \) and \( y'(0) = y'(|\ell|) = 0 \) if \( a = 0 \). Hence the constant function is an eigenfunction corresponding to the eigenvalue 0.

(b) It is easy to verify than an eigenvalue for this problem cannot be negative.

(c) If \( \lambda = \mu^2 > 0 \), then \( y(x) = a \cos \mu x + b \sin \mu x \) and \( y'(0) = 0 \) if \( b = 0 \). Then
\[
y'(\ell) = -a \mu \sin \mu \ell = 0
\]
if
\[
\mu \ell = n\pi, \quad n = 0, \pm 1, \pm 2, \ldots
\]
Hence eigenvalues and eigenfunctions are given by
\[
\lambda_n = \left( \frac{n\pi}{\ell} \right)^2, \quad y_n(x) = \cos \left( \frac{n\pi}{\ell} x \right), \quad n = 0, 1, 2, \ldots
\]

4. \( y(0) + y'(0) = 0, \quad y(\ell) = 0 \)

(a) If \( \lambda = -\mu^2 < 0, \mu > 0 \) then
\[
y(x) = ae^{\mu x} + be^{-\mu x}
\]
and the boundary conditions require that
\[
y(0) + y'(0) = (a + b) + \mu(a - b) = 0
\]
\[
y(\ell) = ae^{\mu \ell} + be^{-\mu \ell} = 0
\]
which implies \( b = \exp(2\mu \ell)a \) and
\[
a[(1 - e^{2\mu \ell}) + \mu(1 + e^{2\mu \ell})] = 0.
\]
This can be written as
\[
e^{2\mu \ell} = \frac{1 - \mu}{1 + \mu}
\]
and by graphing each side it is easy to see that this equation is satisfied only when \( \mu = 0 \). Thus we have \( a = 0 \) and hence \( b = 0 \).

(b) If \( \lambda = 0 \), \( y = ax + b \) and the boundary conditions require that
\[
y(0) + y'(0) = b + a = 0
\]
\[
a(\ell - 1) = 0
\]
If \( \ell = 1 \), then \( a \) is arbitrary and an eigenfunction is \( y(x) = x - 1 \). If \( \ell \neq 1 \), then \( a = 0 \) and hence \( b = 0 \) and so \( \lambda = 0 \) is not an eigenvalue.
(c) If $\lambda = \mu^2 > 0$ then $y = a \cos \mu x + b \sin \mu x$ and to satisfy the boundary conditions we need

$$a + \mu b = 0, \quad a \cos(\mu \ell) + b \sin(\mu \ell) = 0$$

or

$$\tan(\mu \ell) = \mu.$$  

It is easy to see that there are infinitely many eigenvalues $\lambda_n$ that satisfy

$$\sqrt{\lambda_n} = \tan\left(\sqrt{\lambda_n} \ell\right)$$

with corresponding eigenfunctions

$$y_n(x) = \sin\left(\sqrt{\lambda_n} x\right) - \sqrt{\lambda_n} \cos\left(\sqrt{\lambda_n} x\right).$$

5. $y(0) = y(\ell)$, $y'(0) = y'(\ell)$

(a) If $\lambda = -\mu^2 < 0$, then

$$y(x) = a \sinh \mu x + b \cosh \mu x$$

and

$$y(0) = b = y(\ell) = a \sinh(\mu \ell) + b \cosh(\mu \ell)$$

while

$$y'(0) = a\mu = y'(\ell) = a\mu \cosh(\mu \ell) + b\mu \sinh(\mu \ell)$$

or

$$\begin{pmatrix} \sinh(\mu \ell) & \cosh(\mu \ell) - 1 \\ \mu(\cosh(\mu \ell) - 1) & \mu \sinh(\mu \ell) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The determinant of the coefficient matrix is $2\mu(\cosh(\mu \ell) - 1)$ which vanishes only if $\mu = 0$ and so $a = b = 0$.

(b) If $\lambda = 0$, an obvious eigenfunction is $y = 1$.

(c) If $\lambda = \mu^2 > 0$, then

$$y(x) = a \sin(\mu x) + b \cos(\mu x).$$

We see that

$$y(0) - y(\ell) = 0 \quad \text{if} \quad b - (a \sin(\mu \ell) + b \cos(\mu \ell)) = 0$$

and

$$y'(0) - y'(\ell) = 0 \quad \text{if} \quad a\mu - (a\mu \cos(\mu \ell) - b\mu \sin(\mu \ell)) = 0$$

or

$$\begin{pmatrix} -\sin(\mu \ell) & 1 - \cos(\mu \ell) \\ \mu(1 - \cos(\mu \ell)) & \mu \sin(\mu \ell) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We obtain a nontrivial solution if

$$-\mu \sin^2(\mu \ell) - \mu(1 - \cos(\mu \ell))^2 = 0.$$
Expanding the second term and simplifying we arrive at $\cos(\mu \ell) = 1$ and so

$$\mu = \frac{2n\pi}{\ell}, \quad n = \pm 1, \pm 2, \cdots$$

In this case $a, b$ are arbitrary and so to each eigenvalue

$$\lambda_n = \left(\frac{2n\pi}{\ell}\right), \quad n = \pm 1, \pm 2, \cdots$$

there are two linearly independent eigenfunctions

$$y_n(x) = a_n \sin \left(\frac{2n\pi}{\ell} x\right) + b_n \cos \left(\frac{2n\pi}{\ell} x\right)$$

**Theorem 4.7.** The eigenvalues of the Regular Sturm-Liouville Boundary Value Problem, described in Definition 4.1, form a countable, increasing sequence

$$\lambda_1 < \lambda_2 < \cdots, \quad \lim_{n \to \infty} \lambda_n = +\infty.$$ 

For each $\lambda_n$ there is an eigenfunction $\varphi_n$ satisfying

$$\|\varphi_n\| = \left(\int_a^b |\varphi_n(x)|^2 \, dx\right)^{1/2}, \quad \langle \varphi_n, \varphi_m \rangle = \delta_{nm}, \quad n, m = 1, 2, \cdots.$$ 

We call this collection of eigenfunctions an orthonormal system of eigenfunctions.

**Theorem 4.8.** Let $\{\varphi_n\}_{n=1}^{\infty}$ be an orthonormal system of eigenfunctions for a Regular Sturm-Liouville Boundary Value Problem. Let $f, f'$ be piecewise continuous on $[a, b]$, then

$$\frac{f(x^+) + f(x^-)}{2} = \sum_{n=1}^{\infty} \langle f, \varphi_n \rangle \varphi_n, \quad a < x < b.$$ 

If, in addition, $f$ is continuous and satisfies the boundary conditions, then

$$f(x) = \sum_{n=1}^{\infty} \langle f, \varphi_n \rangle \varphi_n, \quad a \leq x \leq b,$$

and the series converges uniformly.

For periodic boundary conditions we have the following result.

**Theorem 4.9.** For a Regular Sturm-Liouville Boundary Value Problem if $f \in L^2(a, b)$ then

$$\left\| f(x) - \sum_{n=1}^{N} \langle f, \varphi_n \rangle \varphi_n(x) \right\| = \left(\int_a^b \left| f(\cdot) - \sum_{n=1}^{N} \langle f, \varphi_n \rangle \varphi_n(\cdot) \right|^2 \, dx\right)^{1/2} \xrightarrow{N \to \infty} 0.$$ 

In this case we say that the sequence

$$f_n(x) = \sum_{n=1}^{N} \langle f, \varphi_n \rangle \varphi_n(x)$$

converges to $f$ in the “mean” or in the sense of $L^2(a, b)$ and we write

$$f = \sum_{n=1}^{\infty} \langle f, \varphi_n \rangle \varphi_n, \quad \text{in } L^2(a, b).$$
4.5 Separation of Variables

Sturm-Liouville Boundary Value Problem arise in many applications to partial differential equations by way of the method of separation of variables. In this section we will investigate several applications of this method for the heat equation wave equation and Laplace’s equation. We also give a brief introduction to the idea of spectral representation of a solution to a dynamical system based on eigenfunction expansions.

Comments on General Spectral Theory

In many applications we have a self-adjoint operator $A$ acting in a separable Hilbert space $H$ with inner product $\langle \cdot, \cdot \rangle$. Also it often happens that the operator $A$ has infinitely many (real) eigenvalues $\lambda_j$ satisfying

$$
\lambda_1 < \lambda_2 < \lambda_3 < \cdots, \quad \lim_{n \to \infty} \lambda_n = \infty,
$$

For each $\lambda_j$ there is a normalized eigenvector $1 - \varphi_j$ (i.e., $A\varphi_j = \lambda_j \varphi_j$) and the eigenvectors $\{\varphi_j\}_{j=1}^\infty$ form a complete orthonormal family in $H$. Here, by a complete orthonormal family we mean that

1. (orthonormal) $\langle \varphi_j, \varphi_k \rangle = \delta_{j,k}$
2. (complete) if $f \in H$ satisfies $\langle f, \varphi_j \rangle = 0$ for all $j$ then $f = 0$.

A consequence of this is that for any $f \in H$ we have

$$
f = \sum_{j=1}^\infty \langle f, \varphi_j \rangle \varphi_j,
$$

and, for

$$
\psi \in \mathcal{D}(A) = \left\{ \psi \in H : \sum_{j=1}^\infty \lambda_j^2 |\langle \psi, \varphi_j \rangle|^2 < \infty \right\}
$$

$$
A\psi = \sum_{j=1}^\infty \lambda_j \langle \psi, \varphi_j \rangle \varphi_j.
$$

In this case we can consider the abstract evolution equation

$$
\frac{d}{dt} y + Ay = 0, \quad y(0) = \varphi \in H.
$$

The solution is given (formally) by

$$
y = e^{-At} \varphi = \sum_{j=1}^\infty e^{-\lambda_j t} \langle \varphi, \varphi_j \rangle \varphi_j.
$$

You can check that formally this works since:
1. First, for $t = 0$ we have $y(0) = \sum_{j=1}^{\infty} \langle \varphi, \varphi_j \rangle \varphi_j = \varphi$

2. Secondly,

$$\frac{dy}{dt} = -\sum_{j=1}^{\infty} \lambda_j e^{-\lambda_j t} \langle \varphi, \varphi_j \rangle \varphi_j$$

and

$$Ay = \sum_{j=1}^{\infty} e^{-\lambda_j t} \langle \varphi, \varphi_j \rangle A\varphi_j = \sum_{j=1}^{\infty} \lambda_j e^{-\lambda_j t} \langle \varphi, \varphi_j \rangle \varphi_j$$

and therefore

$$\frac{dy}{dt} + Ay = 0.$$

**Heat and Wave Equations on Bounded Domains**

Let us suppose that $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial \Omega$.

**Heat Equation on a Bounded Domain**

If $u(x, t)$ represents the temperature in the body $\Omega \subset \mathbb{R}^n$ at a point $x \in \Omega$ at time $t$, then the propagation (or diffusion) of heat is (approximately) described by the heat equation

$$u_t = \Delta u,$$

where

$$\Delta u = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}$$

is the Laplace operator.

Generally we have a bounded domain $\Omega \subset \mathbb{R}^n$ and we consider a time interval $0 \leq t \leq T < \infty$. Thus the data for the problem generally consists of initial data at time $t = 0$ given as $u(x, 0)$ and some type of boundary data on $\partial \Omega \times [0, T]$.

There are three physically motivated BCs often considered:

1. (Dirichlet) $B_d(u) = u(x, t) = 0$ for $(x, t) \in \partial \Omega \times [0, T]$. Thus the temperature on the surface of the body is zero.

2. (Neumann) $B_n(u) = \frac{\partial u}{\partial n}(x, t) = 0$ for $(x, t) \in \partial \Omega \times [0, T]$. In this case the body is insulated, i.e., there is no heat flow in or out of the region.

3. (Radiation) $B_r(u) = \left( \frac{\partial u}{\partial n} + cu \right)(x, t) = 0$. This condition corresponds to Newton’s Law of Cooling in which outside the region $\Omega$ a temperature $u_0(x, t)$ is maintained and the rate of heat flow in or out of the region is proportional to $u(x, t)$.
The method of separation of variables plays a central role. We seek simple solutions of the heat equation in the form

\[ u(x, t) = X(x)T(t) \]

which gives

\[ \frac{\dot{T}(t)}{T(t)} = \frac{\Delta X(x)}{X(x)} = -\lambda. \]

We conclude that

\[ \dot{T}(t) + \lambda T(t) = 0 \]

and

\[ \Delta X(x) + \lambda X(x) = 0, \quad B(X(x))_{x \in \partial \Omega} = 0. \]

The first of these problems, an ordinary differential equation has general solution

\[ T(t) = A \exp(-\lambda t). \]

**Theorem 4.10.** The Laplacian operator \( \Delta \) on a bounded domain \( \Omega \) with smooth boundary \( \partial \Omega \) and any of the three boundary conditions (\( B_{\gamma} \) for \( \gamma = d, n, r \)) admits an orthonormal basis for \( L^2(\Omega) \) consisting of eigenfunctions \( \psi_j(x) \) with associated eigenvalues \( -\lambda_j \) satisfying

\[ \Delta \psi_j + \lambda_j \psi_j = 0, \quad B_\gamma(\psi)(x) = 0 \quad x \in \partial \Omega. \]

Thus \( \langle \psi_j, \psi_k \rangle = \delta_{j,k} \) where \( \langle \cdot, \cdot \rangle \) is the inner product in \( L^2(\Omega) \) given by

\[ \langle f, g \rangle = \int_{\Omega} f(x)\overline{g(x)} \, dx \]

where \( \overline{z} \) denotes the complex conjugate of \( z \).

In particular, for every \( f \in L^2(\Omega) \) we have

\[ f = \sum_{j=1}^{\infty} \langle f, \psi_j \rangle \psi_j, \]

in the sense of \( L^2(\Omega) \).

With this result we can write the solution to

\[ u_t - \Delta u = 0 \quad (x, t) \in \Omega \times (0, T), \]
\[ u(x, 0) = f(x) \quad x \in \Omega, \]
\[ B_\gamma(u)(x, t) = 0 \quad (x, t) \in \partial \Omega \times [0, T]. \]

as

\[ u(x, t) = \sum_{j=1}^{\infty} e^{-\lambda_j t} \langle f, \psi_j \rangle \psi_j(x), \]
One Dimensional Heat Equation

Consider the forced one dimensional heat equation on a finite interval $[0, 1]$:

$$w_t(x, t) - \epsilon w_{xx}(x, t) = 0$$

where $\phi$ and $f(\cdot, t)$ are in $L^2([0, 1])$, the Hilbert space of square integrable functions on $[0, 1]$ with inner product

$$(f, g) = \int_0^1 f(x)g(x) \, dx.$$

The cases $k_0 = k_1 = 0$ and $k_0 = k_1 = \infty$ are special cases referred to as the Neumann and Dirichlet boundary conditions, respectively. These cases will be treated separately first.

**Dirichlet Conditions**

For motivation let us consider the case with no forcing, i.e., $f(x, t) = 0$, and apply the method of separation of variables.

$$w_t(x, t) - \epsilon w_{xx}(x, t) = 0$$

$x \in [0, 1]$, $t > 0$

$$w(0, t) = 0,$$
$$w(1, t) = 0,$$
$$w(x, 0) = \phi(x)$$

We seek solutions in the form

$$w(x, t) = \chi(x)\mathcal{J}(t).$$

As usual we substitute this expression into (4.5.2) and then divide by $\chi(x)\mathcal{J}(t)$ to obtain

$$\frac{\mathcal{J}(t)'}{\mathcal{J}(t)} = \frac{\epsilon \chi(x)''}{\chi(x)} = \lambda \text{ a constant.}$$
Thus we have
\[ \mathcal{T}(t)' - \lambda \mathcal{T}(t) = 0 \quad (4.5.3) \]
\[ \epsilon \mathcal{X}(x)'' - \lambda \mathcal{X}(x) = 0 \quad (4.5.4) \]
\[ \mathcal{X}(0) = \mathcal{X}(1) = 0 \quad (4.5.5) \]
\[ \mathcal{X}(x) = \phi(x). \quad (4.5.6) \]

By considering the various possibilities for the values of \( \lambda \in \mathbb{C} \) it can be shown that \( \lambda \) must be real and negative. Thus we introduce the new variable \( \mu \) by \( \lambda = -\mu^2/\epsilon \) to obtain
\[ \epsilon \mathcal{X}(x)'' + \mu^2 \mathcal{X}(x) = 0 \quad (4.5.7) \]
\[ \mathcal{X}(0) = \mathcal{X}(1) = 0 \quad (4.5.8) \]
\[ \mathcal{X}(x) = \phi(x). \quad (4.5.9) \]

Equations (4.5.7)-(4.5.8) gives a classical Sturm-Liouville problem with solution given as an infinite collection of eigenpairs \( \{X_j(x), \lambda_j\}_{j=1}^\infty \) where \( \lambda_j = -\epsilon \mu_j^2 \) given by
\[ X_j(x) = c_j \sin(j\pi x), \quad \lambda_j = -j^2\pi^2, \quad j = 1, \ldots \infty. \]

That is, \( \mu_j = j\pi \). We consider the \( L^2 \) inner product
\[ <f, g> = \int_0^1 f(x)g(x) \, dx \]
and associated norm
\[ \|f\| = \sqrt{<f, f>}. \]

On computing the \( L^2 \)-norm of \( \mathcal{X}_j \) we obtain the corresponding orthonormal eigenfunctions
\[ \phi_j(x) = \sqrt{2} \sin(j\pi x). \]

That is, we have
\[ <\phi_j, \phi_k> = \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases} \quad (4.5.10) \]

Now equation (4.5.3) is a first order linear equation with solution \( T_j(t) \) for each eigenvalue \( \lambda_j \) give by
\[ T_j(t) = c_j e^{-j^2\pi^2 t}. \]

Thus for every \( N \) we see that
\[ w_N(x, t) = \sum_{j=1}^{N} c_j e^{-j^2\pi^2 t} \phi_j(x) \]
satisfies (4.5.3)-(4.5.8). We pass to the limit as \( N \) goes to infinity and define
\[ w(x, t) = \sum_{j=1}^{\infty} c_j e^{-j^2\pi^2 t} \phi_j(x) \]
and ask that 

\[ \phi(x) = w(x, 0) = \sum_{j=1}^{N} c_j \phi_j(x) \]  

(4.5.11)

in order to satisfy (4.5.9). The question of whether or not this is possible forms part of the rich history of spectral theory and functional analysis. Formally, if we multiply by \( \phi_k(x) \), integrate from 0 to 1, and use the orthogonality conditions (4.5.10) to get

\[ c_j = \langle \phi, \phi_j \rangle \].

Thus the solution to (4.5.2) is given by

\[ w(x, t) = \sum_{j=1}^{\infty} \langle \phi, \phi_j \rangle e^{-\epsilon j^2 \pi^2 t} \phi_j(x) \]  

(4.5.12)

**Neumann Conditions**

In this example we consider the system

\[ w_t(x, t) - \epsilon w_{xx}(x, t) = 0 \]  

(4.5.13)

\[ x \in [0, 1], \ t > 0 \]

\[ w_x(0, t) = 0, \]  

\[ w_x(1, t) = 0, \]  

\[ w(x, 0) = \phi(x) \]

Repeating the above we have the following.

\[ \mathcal{T}(t)' - \lambda \mathcal{T}(t) = 0 \]  

(4.5.14)

\[ \mathcal{X}(x)''' - \lambda \mathcal{X}(x) = 0 \]  

(4.5.15)

\[ \mathcal{X}'(0) = \mathcal{X}'(1) = 0 \]  

(4.5.16)

\[ \mathcal{X}(x) = \phi(x). \]  

(4.5.17)

Equations (4.5.15)-(4.5.16) gives a classical Sturm-Liouville problem with solution given as an infinite collection of eigenpairs \( \{ \mathcal{X}_j(x), \lambda_j \}_{j=0}^{\infty} \) where

\[ \mathcal{X}_0(x) = 1, \ \mathcal{X}_j(x) = \sqrt{2} \cos(j \pi x), \ \lambda_j = -j^2 \pi^2, \ j = 0, 1, \cdots \infty. \]

The corresponding orthonormal eigenfunctions are thus

\[ \phi_0(x) = 1, \ \phi_j(x) = \sqrt{2} \cos(j \pi x), \ j = 1, 2, \cdots. \]

That is, we have

\[ \langle \phi_j, \phi_k \rangle = \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases} \]  

(4.5.18)
Now equation (4.5.14) is a first order linear equation with solution $T_j(t)$ for each eigenvalue $\lambda_j$ given by

$$T_j(t) = c_je^{-j^2\pi^2 t}.$$  

Thus the solution is given by

$$w(x,t) = \langle \phi,1 \rangle + \sum_{j=1}^{\infty} \langle \phi,\phi_j \rangle e^{-\lambda_j^2 \pi^2 t} \phi_j(x).$$  

**Radiation Conditions**

For the radiation boundary conditions we have the heat equation

$$w_t(x,t) - \epsilon w_{xx}(x,t) = 0$$  

$x \in [0,1]$, $t > 0$

$$w_x(0,t) - k_0 w(0,t) = 0,$$

$$w_x(1,t) + k_1 w(1,t) = 0,$$

$$w(x,0) = \phi(x)$$

In this case the separation of variables leads to finding a basis of solutions (for all $\lambda$ in the complex plane) of the equation

$$y''(x) - \lambda y(x) = 0.$$  

Such a basis is given by

$$y_1(x) = \frac{\sin(\mu x)}{\mu}, \quad y_2(x) = \cos(\mu x)$$

where $\lambda = -\mu^2$ and $\Re(\mu) \geq 0$. Thus every eigenfunction can be written as a linear combination of these basis functions. Applying the boundary conditions in (4.5.20) to a linear combination of these basis functions and computing the determinant of the resulting coefficient matrix we obtain the characteristic equation

$$\left(1 - \frac{\mu^2}{k_0 k_1} \right) \frac{\sin(\mu)}{\mu} + \left( \frac{1}{k_0} + \frac{1}{k_1} \right) \cos(\mu) = 0.$$  

(4.5.21)

This equation has infinitely many zeros $\{\mu_j(k)\}_{j=1}^{\infty}$ satisfying

$$(j-1)\pi \leq \mu_j(k) \leq j\pi, \quad j = 1, 2, \cdots$$  

(4.5.22)

providing the closed loop eigenvalues

$$\lambda_j(k) = \mu_j(k)^2.$$  

For $k_0, k_1 \geq 0$, $k_0 + k_1 > 0$, the above inequalities are strict and from (4.5.21) it is easy to see that

$$\left(\lambda_j(k) - (j\pi)^2\right) \to 0, \quad k_0, k_1 \to \infty, \quad j = 1, 2, \cdots.$$  

(4.5.23)
Corresponding to the eigenvalues $\lambda_j(k) = \mu_j^2(k)$, there is a complete orthonormal system of eigenfunctions in $L^2(0,1)$ given by

$$\psi_j^k(x) = \kappa_j(k) \sin(\mu_j(k)x + \theta_j(k)), \quad j = 1, 2, \ldots,$$

where in (4.5.24)

$$\sin(\theta_j(k)) = \frac{\mu_j(k)}{\sqrt{k_0^2 + \mu_j(k)^2}}, \quad \cos(\theta_j(k)) = \frac{k_0}{\sqrt{k_0^2 + \mu_j(k)^2}},$$

and $\kappa_j(k)$ is a normalization constant given by

$$\kappa_j(k) = \sqrt{\frac{2a_0a_1}{a_0a_1 + (1/k_0 + 1/k_1)c}}$$

with

$$a_0 = 1 + \mu_j(k)^2/k_0^2, \quad a_1 = 1 + \mu_j(k)^2/k_1^2, \quad c = 1 + \mu_j(k)^2/(k_0k_1).$$

Thus the solution to (4.5.20) is given by

$$w(x,t) = \sum_{j=1}^{\infty} \langle \phi, \psi_j^k \rangle e^{\lambda_j t} \psi_j^k(x)$$

**Wave Equation on a Bounded Domain**

The vibrations of a bounded region $\Omega$ in $\mathbb{R}^n$ are governed by a hyperbolic partial differential equation called the wave equation. When we solve the wave equation in domains with boundaries we must specify data on the boundary of the physical domain. The most common boundary conditions are the Dirichlet and Neumann conditions. These led to the following Initial Boundary Value Problems (IBVP)

1. (Dirichlet BC)

$$u_{tt} = \Delta u, \quad x \in \Omega, \quad t \in \mathbb{R}, \quad (4.5.27)$$

**IC** $u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega$

**BC1** $u(x,t) = 0, \quad x \in \partial\Omega, \quad t \in \mathbb{R}, \quad (4.5.28)$

2. (Neumann BC)

$$u_{tt} = \Delta u, \quad x \in \Omega, \quad t \in \mathbb{R}, \quad (4.5.29)$$

**IC** $u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega$

**BC2** $\frac{\partial u}{\partial n}(x,t) = 0, \quad x \in \partial\Omega, \quad t \in \mathbb{R}, \quad (4.5.30)$

where $\frac{\partial u}{\partial n}$ denotes the normal derivative to the boundary of $\Omega$. 
Just as in the case of the heat equation, we rely on the method of separation of variables and spectral theory of the Laplace operator. Once again we rely on the validity of a general result from functional analysis for the Laplacian operator on bounded domains with smooth boundary.

**Theorem 4.11.** The Laplacian operator $\Delta$ on a bounded domain $\Omega$ with smooth boundary $\partial \Omega$ admits an orthonormal basis for $L^2(\Omega)$ consisting of eigenfunctions $\psi_j(x)$ with associated eigenvalues $(-\lambda_j^2) < 0$ satisfying

$$\Delta \psi_j + \lambda_j^2 \psi_j = 0.$$ 

Thus $\langle \psi_j, \psi_k \rangle = \delta_{j,k}$ where $\langle \cdot, \cdot \rangle$ is the inner product in $L^2(\Omega)$ given by

$$\langle f, g \rangle = \int_{\Omega} f(x) \overline{g(x)} \, dx$$

where $\overline{z}$ denotes the complex conjugate of $z$.

In particular, for every $f \in L^2(\Omega)$ we have

$$f = \sum_{j=1}^{\infty} \langle f, \psi_j \rangle \psi_j,$$

in the sense of $L^2(\Omega)$.

With this result we can write the solution to (4.5.27) - (4.5.28) or (4.5.29) - (4.5.30) as

$$u(x, t) = \sum_{j=1}^{\infty} \left( a_j \cos(\lambda_j t) + b_j \sin(\lambda_j t) \right) \psi_j(x),$$

where

$$u_0(x) = \sum_{j=1}^{\infty} a_j \psi_j(x), \quad u_1(x) = \sum_{j=1}^{\infty} \lambda_j b_j \psi_j(x),$$

and

$$a_j = \langle u_0, \psi_j \rangle, \quad b_j = \lambda_j^{-1} \langle u_1, \psi_j \rangle.$$

**Finite Vibrating String**

The transverse displacement of a string of length $\ell$ that is fixed at the end points is determined by the system

$$u_{tt} = c^2 u_{xx}, \quad 0 < x < \ell, \quad t \in \mathbb{R},$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad 0 \leq x \leq \ell,$$

$$u(0, t) = 0, \quad u(\ell, t) = 0, \quad t \geq 0.$$ 

We use the method of separation of variables to analyze this problem. For this method we seek simple solutions in the form $u(x, t) = X(x)T(t)$. Plugging this expression into the equation we have

$$X \ddot{T} = c^2 X'' T$$
or
\[
\frac{X''}{X} = \frac{\ddot{T}}{c^2 T}.
\]
Since the left side depends only on \(x\) and the right side depends only on \(t\) it follows that the expressions must be a constant, say \(-\lambda\) and so we obtain
\[
\ddot{T} + c^2 \lambda T = 0
\]
and
\[
X'' + \lambda X = 0,
\]
\[
X(0) = 0 = X(\ell).
\]
This is a Sturm-Liouville two point boundary value problem as we considered earlier. Recall that such a problem has infinitely many values of \(\lambda\) (called eigenvalues) and associated eigenfunctions.

In particular we have eigenvalues
\[
\lambda = \lambda_n^2 = \frac{n^2 \pi^2}{\ell^2}, \quad n = 1, 2, 3, \ldots
\]
and associated eigenfunctions
\[
X_n(x) = C_n \sin \left( \frac{n\pi x}{\ell} \right).
\]

Now for each \(n\) we can solve for functions \(T_n\) to get
\[
T_n(t) = A_n \cos(\lambda_n ct) + B_n \sin(\lambda_n ct).
\]
Thus we obtain a family of simple solutions
\[
u_n(x, t) = [A_n \cos(\lambda_n ct) + B_n \sin(\lambda_n ct)] \sin(\lambda_n x).
\]
Each of these functions satisfies the wave equation and the boundary conditions in (4.5.31). In order to satisfy the initial conditions we seek constants \(\{a_n\}\) and \(\{b_n\}\) so that the solution to the full system (4.5.31) is given by
\[
\begin{aligned}
  u(x, t) &= \sum_{n=1}^{\infty} u_n(x, t) = \sum_{n=1}^{\infty} [a_n \cos(\lambda_n ct) + b_n \sin(\lambda_n ct)] \varphi_n(x), \\
  u_0(x) &= u(x, 0) = \sum_{n=1}^{\infty} a_n \varphi_n(x), \quad 0 < x < \ell,
\end{aligned}
\]
where
\[
\varphi_n(x) = \sqrt{\frac{2}{\ell}} \sin(\lambda_n x).
\]
To satisfy the initial conditions we need
\[
\begin{aligned}
  u_0(x) &= u(x, 0) = \sum_{n=1}^{\infty} a_n \varphi_n(x), \quad 0 < x < \ell,
\end{aligned}
\]
and
\[ u_1(x) = u_t(x, 0) = \sum_{n=1}^{\infty} (\lambda_n c) b_n \varphi_n(x), \quad 0 < x < \ell. \]  
(4.5.34)

Finally
\[ a_n = \langle u_0, \varphi_n \rangle, \quad a_n = \frac{1}{c\lambda_n} \langle u_1, \varphi_n \rangle, \quad n = 1, 2, \ldots. \]

**Vibrating Rectangular Membrane**

Consider the following problem for a two dimensional wave equation
\[ u_{tt} = u_{xx} + u_{yy}, \quad 0 < x < a, \quad 0 < y < b, \quad t \in \mathbb{R}, \]  
(4.5.35)
\[ u(x, y, 0) = u_0(x, y), \quad u_t(x, y, 0) = u_1(x, y), \quad 0 \leq x \leq a, \quad 0 \leq y \leq b, \]
\[ u(0, y, t) = u(a, y, t) = u(x, 0, t) = u(x, b, t) = 0, \quad t \in \mathbb{R}. \]

We again seek a solution using the method of *separation of variables*. In this case we seek simple solutions in the form \( u(x, y, t) = v(x, y)T(t) \). Plugging this expression into the equation we have
\[ v\ddot{T} = \Delta v T \]
or
\[ \frac{\Delta v}{v} = \frac{\dot{T}}{T} = -\lambda^2. \]

We obtain
\[ \ddot{T} + \lambda^2 T = 0 \]
and
\[ \Delta v + \lambda^2 v = 0, \]
\[ v \mid_{(x,y) \in \partial \Omega} = 0. \]

In order to solve this last problem we again use separation of variables to seek \( v(x, y) = X(x)Y(y) \). This leads to
\[ -\frac{X''}{X} = \frac{Y''}{Y} + \lambda^2 = \mu^2 \]
or, with \( \nu^2 = \lambda^2 - \mu^2 \)
\[ X'' + \mu X = 0 \quad \quad Y'' + \nu^2 Y = 0 \]
\[ X(0) = X(a) \quad \quad Y(0) = Y(b). \]

Thus we obtain
\[ X_m(x) = D_m \sin(\mu_m x), \quad \mu_m^2 = \frac{m^2 \pi^2}{a^2} \]
\[ Y_n(y) = E_n \sin(\nu_n y), \quad \nu_n^2 = \frac{n^2 \pi^2}{b^2}. \]
From this we see that the eigenvalue problem
\[ \Delta v + \lambda^2 v = 0, \text{in } \Omega = [0, a] \times [0, b], \quad v|_{\partial \Omega} \]
has eigenvalues
\[ \lambda_{n,m}^2 = \pi^2 \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right), \quad n, m = 1, 2, \cdots. \]

It is worth noting briefly that interesting number theoretic questions arose in these problems.

Returning to the two dimensional wave equation we find an infinite number of solutions of the form
\[ u_{m,n} = \sin \left( \frac{m \pi x}{a} \right) \sin \left( \frac{n \pi y}{b} \right) \left( a_{m,n} \cos (\lambda_{m,n} t) + b_{m,n} \sin (\lambda_{m,n} t) \right). \]

Just as in the last example we seek a solution to the main problem (4.5.35) in the form
\[ u(x, t) = \sum_{m,n=1} u_{m,n}(x, y, t). \]

Let us define
\[ \psi_m(x) = \sqrt{\frac{2}{a}} \sin \left( \frac{m \pi x}{a} \right), \quad \varphi_n(y) = \sqrt{\frac{2}{b}} \sin \left( \frac{n \pi y}{b} \right). \]

In order to satisfy the initial conditions we choose \( a_{m,n}, b_{m,n} \) so that
\[ u_0(x, y) = \sum_{m,n=1} a_{m,n} \psi_m(x) \varphi_n(y) \]
\[ u_1(x, y) = \sum_{m,n=1} \lambda_{m,n} b_{m,n} \psi_m(x) \varphi_n(y). \]

There is of course a corresponding theory of fourier series expansions for higher dimensional Fourier series, (see for example [2] for an excellent disscussion of this whole block of material on Fourier series). We note only that the Fourier coefficients are given by
\[ a_{m,n} = \int_0^a \int_0^b u_0(x, y) \psi_m(x) \varphi_n(y), \quad b_{m,n} = \int_0^a \int_0^b u_1(x, y) \psi_m(x) \varphi_n(y) \]
\[ \sqrt{\lambda_{m,n}} b_{m,n} = \int_0^a \int_0^b u_1(x, y) \psi_m(x) \varphi_n(y) \quad dx \quad dy. \]

A famous problem in elementary number theory is to show that the multiplicity of the expressions \( n^2 + m^2 \) actually grows without bound, i.e., the multiplicity of the eigenvalues becomes infinite. It is only in the case that \( a \) and \( b \) are irrational and \( a/b \) is also irrational, that the eigenvalues all have multiplicity one.

Consider for a moment the case in which \( a = b \) so that
\[ \lambda_{m,m} = \frac{\pi^2}{a^2} \left( m^2 + n^2 \right), \quad n, m = 1, 2, \cdots. \]
In this case a natural question arises. What is the multiplicity of a given eigenvalue, that is, how many ways can an integer be expressed as the sum of squares of two integers? Consider for example

<table>
<thead>
<tr>
<th>$(m, n)$</th>
<th>$\lambda_{m,n}$</th>
<th>$\varphi_{m,n}(x, y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1, 1)$</td>
<td>$\frac{\pi}{a}\sqrt{2}$</td>
<td>$\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{a}\right)$</td>
</tr>
<tr>
<td>$(2, 1)$</td>
<td>$\frac{\pi}{a}\sqrt{5}$</td>
<td>$\sin\left(\frac{2\pi x}{a}\right)\sin\left(\frac{\pi y}{a}\right)$</td>
</tr>
<tr>
<td>$(1, 2)$</td>
<td>$\frac{\pi}{a}\sqrt{5}$</td>
<td>$\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi y}{a}\right)$</td>
</tr>
<tr>
<td>$(2, 2)$</td>
<td>$\frac{\pi}{a}\sqrt{8}$</td>
<td>$\sin\left(\frac{2\pi x}{a}\right)\sin\left(\frac{2\pi y}{a}\right)$</td>
</tr>
<tr>
<td>$(3, 1)$</td>
<td>$\frac{\pi}{a}\sqrt{10}$</td>
<td>$\sin\left(\frac{3\pi x}{a}\right)\sin\left(\frac{\pi y}{a}\right)$</td>
</tr>
<tr>
<td>$(1, 3)$</td>
<td>$\frac{\pi}{a}\sqrt{10}$</td>
<td>$\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{3\pi y}{a}\right)$</td>
</tr>
</tbody>
</table>

**Stationary Heat and Wave Equations – Elliptic Problems**

Many problems that involve elliptic equations arise as steady state problems for what was originally a time dependent problem. That is we seek a solution of a time dependent problem that happen to be independent of time. Such a problem arises for example in the study of steady state temperature distribution in a domain $\Omega \subset \mathbb{R}^n$ given the temperature distribution on the boundary $\partial\Omega$ of $\Omega$. In this case the so-called Dirichlet problem is find a function $u(x)$ satisfying

$$\Delta u = 0, \quad x \in \Omega \subset \mathbb{R}^n,$$
$$u(x) = f(x), \quad x \in \partial\Omega.$$ 

If, instead of the temperature, we are given the flux on the boundary, then we need to solve the so called Neumann Problem

$$\Delta u = 0, \quad x \in \Omega \subset \mathbb{R}^n,$$
$$\frac{\partial u}{\partial \nu}(x) = f(x), \quad x \in \partial\Omega. $$

where $\frac{\partial u}{\partial \nu}(x)$ represents the normal derivative to the boundary of $\Omega$. 

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One final example of a common elliptic boundary value problem that arises in applications is the so called Robin Problem

\[ \Delta u = 0, \quad x \in \Omega \subset \mathbb{R}^n, \]
\[ \frac{\partial u}{\partial \nu}(x) + \alpha u(x) = \beta, \quad x \in \partial \Omega. \]

**Example 4.3 (Dirichlet Problem for a Square).** We use the method of separation of variables to solve the Dirichlet Problem

\[ \Delta u = 0, \quad 0 \leq x \leq \pi, \quad 0 \leq y \leq a, \]
\[ u(0, y) = 0, \quad u(\pi, y) = 0, \quad 0 \leq y \leq a, \]
\[ u(x, 0) = f(x), \quad u(x, a) = 0. \]

We seek simple solutions of Laplace’s equation in the form \( u(x, y) = X(x)Y(y) \) which leads to

\[ \frac{X''}{X} = -\frac{Y''}{Y} = -\lambda, \]

and thus we obtain

\[ X'' + \lambda X = 0 \]
\[ X(0) = X(\pi) = 0. \]

We have a nontrivial solution if

\[ \lambda_n = n^2, \quad X_n(x) = c_n \sin(nx), \quad n = 1, 2, \ldots. \]

For \( Y(y) \) we have

\[ Y''(y) - n^2Y(y) = 0 \]
\[ Y(a) = 0. \]

Thus, for convenience, we can take \( Y_n(y) = \frac{\sinh[n(a - y)]}{\sinh(na)} \). We must still satisfy the boundary condition at \( y = 0 \) and we seek a formal solution in the

\[ u(x, y) = \sum_{n=1}^{\infty} c_n \frac{\sinh[n(a - y)]}{\sinh(na)} \sin(nx) \]

and

\[ f(x) = u(x, 0) = \sum_{n=1}^{\infty} c_n \sin(nx) \]

\[ (*) \]
from which we know that
\[
c_n = \frac{2}{\pi} \int_{0}^{\pi} f(x) \sin(nx) \, dx.
\]

We still need to show that this formal solution is actually a solution. To this end we quote a theorem from advanced calculus concerning the convergence of a series of products. Either of the following two results can be used to do the proof. The first is due to Dirichlet and the second is usually referred to as Abel’s Theorem.

**Theorem 4.12 (Dirichlet’s Test).** Let \( \sum_{n=1}^{\infty} a_n \) be a series with bounded partial sums. Let \( \{b_n\} \) be a monotonic sequence converging to zero. Then the series \( \sum_{n=1}^{\infty} a_n b_n \) converges.

**Proof.** Let \( A_n = a_1 + a_2 + \cdots + a_n \) and assume that \( A_n \leq M \) for all \( n \). Then \( \lim_{n \to \infty} A_n b_{n+1} = 0 \), then by the summation by parts formula
\[
\sum_{n=1}^{n} a_n b_n = A_n b_{n+1} - \sum_{k=1}^{n} A_k (b_{k+1} - b_k),
\]
to show the series converges we need only show that \( \sum A_k (b_{k+1} - b_k) \) converges. Since the terms \( b_n \) are decreasing we have
\[
|A_k (b_{k+1} - b_k)| \leq M (b_{k+1} - b_k).
\]
But the series \( \sum (b_{k+1} - b_k) \) is a convergent telescoping series. Thus by the comparison test we obtain *absolute* convergence of series \( \sum A_k (b_{k+1} - b_k) \).

**Theorem 4.13 (Abel’s Theorem).** The series \( \sum_{n=1}^{\infty} a_n b_n \) converges if \( \sum_{n=1}^{\infty} a_n \) converges and \( \{b_n\} \) is a monotonic convergent sequence.

**Proof.** The convergence of \( \sum_{n=1}^{\infty} a_n b_n \) and \( \{b_n\} \) proves the convergence of \( A_n b_{n+1} \) where, as above, \( A_n = a_1 + a_2 + \cdots + a_n \). Also, \( \{A_n\} \) is a bounded sequence. The remainder of the proof goes like the proof of Theorem 4.12.

As a consequence of Abel’s Theorem, we state without proof, a result which directly yields our desired result.

**Theorem 4.14.** The series \( \sum_{n=1}^{\infty} X_n(x)Y_n(y) \) converges uniformly in a set \( I \times J \) in the plane provided

1. \( \sum X_n(x) \) converges uniformly for \( x \in I \).

2. The sequence \( \{Y_n(y)\} \) is uniformly bounded and convergent for \( y \in J \) and monotonic with respect to \( n \) for all \( y \in J \).
In our case we note that if \( f(0) = f(\pi) = 0 \) (which should be true by continuity) and \( f \) is continuous and piecewise smooth, then by our earlier result from Chapter 7, we know that the series (*) for \( f \) converges for every \( x \). So the first condition of Theorem 4.14 is satisfied. Next we note that

\[
Y_n(y) = \frac{e^{n(a-y)} - e^{-n(a-y)}}{e^{na} - e^{-na}} \\
= e^{-ny} \frac{1 - e^{-2n(a-y)}}{1 - e^{-2na}} \\
\leq \frac{e^{-ny}}{1 - e^{-2na}} \leq \frac{e^{-ny}}{1 - e^{-2a}}.
\]

By this esitmate we have

\[
Y_n(y) \leq \frac{1}{1 - e^{-2a}}
\]

which gives uniform boundedness and, for each \( y \)

\[
Y_n(y) = \frac{e^{n(a-y)} - e^{-n(a-y)}}{e^{na} - e^{-na}}
\]

is a monotonic convergent sequence in \( n \) for each \( y \). Thus the second condition in Theorem 4.14 is satisfied.

Clearly \( u(0, y) = u(\pi, y) = 0 \) and \( u(x, a) = 0 \). We also note that for \( u_x, u_{xx}, u_y, u_{yy} \) the series of differentiated terms are dominated by

\[
C \sum_{n=1}^{\infty} n^2 e^{-ny}
\]

and so term-by-term differentiation is justified and we see that \( u \) is the solution to our Dirichlet Problem.

### 4.6 Green’s Functions

One of the most common ways to establish the validity of the expansion theorems for Regular Sturm-Liouville Boundary Value Problem given in Theorems 4.7, 4.8, 4.9 is using spectral theory of compact self-adjoint operators in Hilbert space. Due to time constraints we are not in a position to develop these results. But the first step is to write the solution of a nonhomogeneous second order problem in terms of an integral operator whose kernel is called a Green’s function. Namely, we consider a nonhomogeneous problem

\[
L \varphi = f, \quad f \in L^2[a, b]
\]

where

\[
L = \frac{d}{dx} \left( p(x) \frac{d}{dx} \right) + q,
\]
\[ B_1(\varphi) = \alpha_1 \varphi(a) + \alpha_2 \varphi'(a) = 0 \quad (4.6.1) \]
\[ B_2(\varphi) = \beta_1 \varphi(b) + \beta_2 \varphi'(b) = 0. \quad (4.6.2) \]

We will construct a function \( G(x,y) \) so that the solution is given by

\[ \varphi(x) = \int_a^b G(x,y) f(y) \, dy. \]

**Definition 4.3.** A Green’s function for \( L \) with boundary conditions (4.6.1) and (4.6.2) is a function \( G(x,\xi) \) for \((x,\xi) \in [a,b] \times [a,b]\) such that

1. The following hold
   (a) \( G(\cdot,\cdot) \) is continuous on \([a,b] \times [a,b]\),
   (b) \( \frac{\partial G}{\partial x}(\cdot,\xi) \) is continuous on \([a,\xi) \times (\xi,b]\), and,
   (c) \( \left. \frac{\partial G(x,\xi)}{\partial x} \right|_{x=\xi^+} \equiv \frac{\partial G}{\partial x}(\xi^+,\xi,\lambda) - \frac{\partial G}{\partial x}(\xi^-,\xi,\lambda) = \frac{1}{p(\xi)} \)

2. for all \( \xi \in [a,b] \), \( G(x,\xi) \) solves \( L(G) = 0, \quad x \neq \xi \).

3. for all \( \xi \in (a,b) \), \( B_i(G) = 0 \).

**Theorem 4.15.** Under the assumption that

\[ L(y) = 0, \quad B_1(y) = 0, \quad B_2(y) = 0 \quad \text{implies} \quad y \equiv 0 \]

(i.e., the only solution of the homogeneous problem is the zero solution), there is a unique Green’s function \( G(x,\xi) \). Furthermore \( G(x,\xi) \) is symmetric, i.e., \( G(x,\xi) = G(\xi,x) \).

**Proof.** We provide a proof by construction

\[ B_1(y) = \alpha_1 y(a) + \alpha_2 y'(a) = 0 \]
\[ B_2(y) = \beta_1 y(b) + \beta_2 y'(b) = 0. \]

Choose \( u_i \) such that \( L(u_i) = 0 \) and \( B_i(u_i) = 0 \). This can be done it amounts to solving an initial value problem corresponding to initial conditions specified at \( x = a \) and \( x = b \). For example we could choose \( u_1 \) so that \( u_1(a) = \alpha_2 \) and \( u'_1(a) = -\alpha_1 \). By the fundamental existence uniqueness theorem such a solution exists. The solutions \( u_1, u_2 \) must be linearly independent. Indeed, suppose \( w = c_1 u_1 + c_2 u_2 \equiv 0 \). Then

\[ B_1(w) = c_2 B_1(u_2) = 0 \]
\[ B_2(w) = c_1 B_2(u_1) = 0. \]

If \( B_1(u_2) = 0 \) then \( u_2 \) would be a solution of the homogeneous problem which we assumed only had the zero solution. Thus \( u_2 \) would be zero which it is not. Hence we must have \( c_2 = 0 \). Similarly, \( c_1 = 0 \).
Now seek $G(x, \xi)$ in the form

$$G(x, \xi) = \begin{cases} Au_1(x) & a \leq x \leq \xi \\ Bu_2(x) & \xi \leq x \leq b. \end{cases}$$

We need

$$Au_1(\xi) = Bu_2(\xi)$$

and

$$Bu'_2(\xi) - Au'_1(\xi) = \frac{1}{p(\xi)}.$$

By Cramer’s rule, one obtains

$$A = \frac{u_2(\xi)}{p(\xi)W(u_1, u_2)(\xi)}$$

$$B = \frac{u_1(\xi)}{p(\xi)W(u_1, u_2)(\xi)}$$

and hence

$$G(x, \xi) = \begin{cases} \frac{u_1(x)u_2(\xi)}{p(\xi)W(u_1, u_2)(\xi)} & a \leq x \leq \xi \\ \frac{u_1(\xi)u_2(x)}{p(\xi)W(u_1, u_2)(\xi)} & \xi \leq x \leq b. \end{cases}$$

To see that $G$ is symmetric, recall that Abel’s formula states that for the equation

$$y'' + \frac{p'}{p}y' + \frac{q}{p}y = 0$$

the Wronskian satisfies

$$W(u_1, u_2)(\xi) = W(x_0) \exp \left( - \int_{x_0}^{\xi} \frac{p'(x)}{p(x)} \, dx \right) = W(x_0) \frac{p(x_0)}{p(\xi)}.$$ 

Hence

$$W(u_1, u_2)(\xi)p(\xi) = \text{constant}.$$ 

We now turn to the main application of Green’s function in this section. Namely, we consider the nonhomogeneous BVP.

$$L(y) = (py')' + q(x)y = f(x), \quad a < x < b$$

$$B_1(y) = 0 \quad (4.6.3)$$

$$B_2(y) = 0 \quad (4.6.4)$$

with $p \in C^1(a, b), p(x) > 0, x \in [a, b]$. 

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First we recall a classical formula whose general counterpart has far reaching consequences in the theory of ordinary and partial differential equations and the theory of weak solutions. At this point we will only consider a very special case. Namely, given any two functions \( u \) and \( v \), a straightforward calculation gives the so-called Lagrange Identity:

\[
vL(u) - uL(v) = \frac{d}{dx} P(u, v)
\]

where (see (??))

\[
P(u, v) = p(u'v - uv')
\]

and we note that integration gives the Green’s formula

\[
\int_a^b [vL(u) - uL(v)] = P(u, v) \bigg|_{x=a}^{x=b}.
\]

Let \( G(x, \xi) \) denote the Green’s function for the homogeneous BVP \( \lambda \). From Lagrange’s identity, for \( x \neq \xi \)

\[
G(x, \xi) L(y) - yL(G(x, \xi)) = \frac{d}{dx} [p(Gy' - yG')]
\]

which implies

\[
\int_a^{\xi^-} GL(y) dx = p(Gy' - G' y)|_{\xi^-}^a
\]

and

\[
\int_{\xi^+}^b GL(y) dx = p(Gy' - G' y)|_{\xi^+}^b.
\]

Hence

\[
\int_a^b GL(y) dx = p(Gy' - G' y)|_a^b - p(Gy' - G' y)|_{\xi^-}^{\xi^+}.
\]

For our boundary condition \( B_1, B_2 \) we have

\[
[p(Gy' - G' y)]_a^b = 0.
\]

Thus

\[
\int_a^b GL(y) dx = -[p(Gy' - G' y)]_{\xi^-}^{\xi^+}
\]

\[
= p \left[ \frac{\partial G}{\partial x} (\xi^+, \xi) - \frac{\partial G}{\partial x} (\xi^-, \xi) \right] y(\xi)
\]

\[
= y(\xi).
\]

Thus, formally at least, if \( y \) satisfies \( L(y) = f \), then we should have \( y(x) = \int_a^b G(x, \xi, \lambda) f(\xi) d\xi \).

That is, on a purely formal level, we have

\[
\int_a^b G(x, \xi) L(y)(x)dx = \int_a^b LG(x, \xi)(y)(x)dx = y(\xi)
\]
which suggest that \( LG(x, \xi) = \delta(x - \xi) \) the dirac delta function, i.e., the solution to
\[
L(y) = f \\
B_i(y) = 0
\]
would be given by
\[
y(x) = \int_a^b G(x, \xi) f(\xi) d\xi
\]
provided that the only solution to the homogeneous problem is the zero function.

**Theorem 4.16.** If the only solution to the homogeneous problem is the zero function, then the unique solution of
\[
Ly = f \\
B_1(y) = \gamma_1, B_2(y) = \gamma_2
\]
is given by
\[
y(x) = \frac{\gamma_2}{B_2(y_1)} y_1(x) + \frac{\gamma_1}{B_1(y_2)} y_2(x) + \int_a^b G(x, \xi) f(\xi) d\xi
\]
where \( y_1, y_2 \) are (not identically zero) solutions of
\[
L(y) = 0 \\
B_1(y_1) = 0, \quad B_2(y_2) = 0.
\]
(Note that since \( y_1 \) and \( y_2 \) are not identically zero, we must have \( B_1(y_2) \neq 0 \) and \( B_2(y_1) \neq 0 \).

**Proof.** Since \( B_1(G) = B_2(G) = 0 \),
\[
B_1(y) = \frac{\gamma_1}{B_1(y_2)} B_1(y_2) = \gamma_1
\]
and similarly \( B_2(y) = \gamma_2 \).

To see that the nonhomogeneous differential equation is satisfied, we consider
\[
u(x) = \int_a^b G(x, \xi) f(\xi) d\xi
\]
\[
= \int_a^x G(x, \xi) f(\xi) d\xi + \int_x^b G(x, \xi) f(\xi) d\xi
\]
Then
\[
u'(x) = \int_a^x \frac{\partial G}{\partial x} f d\xi + G(x, x^-) f(x^-)
\]
\[
+ \int_x^b \frac{\partial G}{\partial x} f d\xi - G(x, x^+) f(x^+)
\]
\[
= \int_a^x G_x(x, \xi) f(\xi) d\xi + \int_x^b G_x(x, \xi) f(\xi) d\xi.
\]
Differentiating again we have
\[
u''(x) = \int_a^x G_{xx}(x, \xi) f(\xi) d\xi + G_x(x, x^-) f(x^-)
\]
\[
+ \int_x^b G_{xx}(x, \xi) f(\xi) d\xi - G_x(x, x^+) f(x^+).
\]
We need the following observation,

\[
\frac{\partial G}{\partial x}(x, x^-) = \frac{\partial G}{\partial x}(x^+, x)
\]

\[
\frac{\partial G}{\partial x}(x, x^+) = \frac{\partial G}{\partial x}(x^-, x).
\]

For example, to verify the first of these claims, note that

\[
\frac{\partial G}{\partial x}(x, x^-) = \lim_{\epsilon \to 0^+} \frac{G(x + h, x - \epsilon) - G(x, x - \epsilon)}{h}
\]

The partials exist because we are in the open region \(x > \xi\) away from the diagonal \((x = \xi)\) where only one-sided derivatives exist. Moreover, because \(G\) is smooth when \(x > \xi\) we may interchange the order of limits to obtain

\[
\frac{\partial G}{\partial x}(x, x^-) = \lim_{h \to 0^+} \lim_{\epsilon \to 0^+} \frac{G(x + h, x - \epsilon) - G(x, x - \epsilon)}{h}
\]

Similarly for the other statement. Hence we have

\[
\begin{align*}
u''(x) &= \int_a^b G_{xx}(x, \xi) f(\xi) d\xi + [G_x(x^+, x) - G_x(x^-, x)] f(x^-) \\hline&= \int_a^b G_{xx}(x, \xi) f(\xi) d\xi + f(x)/p(x).\end{align*}
\]

Thus

\[
L(u) = pu'' + p'u' + qu
\]

\[
= \int_a^b \left[p(x)G_{xx}(x, \xi) + p'(x)G_x(x, \xi) + q(x)G(x, \xi)\right] f(\xi) d\xi + \frac{p(x)f(x)}{p(x)}
\]

\[
= \int_a^b LG(x, \xi)y(\xi) d\xi + f(x)
\]

\[
= f(x)
\]

since \(L(G) = 0\).
Example 4.4. Find the Green’s function for
\[ y'' = 0, \quad 0 < x < 1 \]
\[ y(0) = 0, \quad y(1) = 0. \]

Here \( L = \frac{d^2}{dx^2} \) and it is easy to verify that the only solution of the homogeneous problem is the zero solution. Take \( u_1(x) = x \), \( u_2(x) = x - 1 \) and
\[ W(u_1, u_2) = \begin{vmatrix} x & x - 1 \\ 1 & 1 \end{vmatrix} = 1 \]
Hence
\[ G(x, \xi) = \begin{cases} x(\xi - 1) & 0 \leq x \leq \xi \\ \xi(x - 1) & \xi \leq x \leq 1 \end{cases} \]

Example 4.5. Find the Green’s function for
\[ y'' + \lambda y = 0, \quad \lambda > 0, \quad 0 < x < \pi \]
\[ y(0) = 0, \quad y(\pi) = 0. \]

Here we regard \( L(y) = y'' + \lambda y \). We know, from our earlier work, that the eigenvalues are given by
\[ \lambda_n = n^2, \quad n = 1, 2, 3 \ldots \]
so the only solution of the homogeneous problem is the zero solution. If \( \lambda \neq n \), take
\[ u_1(x) = \sin(\sqrt{\lambda}x), \quad u_2(x) = \sin(\sqrt{\lambda}(x - \pi)). \]
Then
\[ p(0)W(u_1, u_2)(0) = \begin{vmatrix} 0 & \sin(\sqrt{\lambda}\pi) \\ \sqrt{\lambda} & \sqrt{\lambda} \cos(\sqrt{\lambda}\pi) \end{vmatrix} = -\sqrt{\lambda}\sin(\sqrt{\lambda}\pi) \]
and so
\[ G(x, \xi, \lambda) = \begin{cases} -\sin(\sqrt{\lambda}x)\sin(\sqrt{\lambda}(\xi - \pi)) / \sqrt{\lambda}\sin(\sqrt{\lambda}\pi) & 0 \leq x \leq \xi \\ -\sin(\sqrt{\lambda}\xi)\sin(\sqrt{\lambda}(x - \pi)) / \sqrt{\lambda}\sin(\sqrt{\lambda}\pi) & \xi \leq x \leq 1 \end{cases} \]

The next example does not exactly fit the program developed above but the same ideas work anyway.

Example 4.6. This example corresponds to the solution of nonhomogeneous Initial Value Problems. That is the boundary conditions are actually the initial conditions
\[ B_1(u) = u(a) \]
\[ B_2(u) = u'(a) \]
In this case we seek

\[ G(x, \xi, \lambda) = \begin{cases} 0 & a \leq x \leq \xi \\ Au_1(x) + Bu_2(x) & \xi \leq x \end{cases} \]

where \( u_1, u_2 \) are linearly independent solutions of \( L_\lambda = 0 \). In this case the continuity and jump condition give

\[ Au_1(\xi) + Bu_2(\xi) = 0 \]

and

\[ Au_1'(\xi) + Bu_2'(\xi) = \frac{1}{p(\xi)}. \]

and so

\[ A = -\frac{u_2(\xi)}{k(\xi)W(u_1, u_2)(\xi)}, \quad B = \frac{u_1(\xi)}{k(\xi)W(u_1, u_2)(\xi)}. \]

Hence

\[ G(x, \xi, \lambda) = \begin{cases} 0 & a \leq x \leq \xi \\ \frac{u_1(\xi)u_2(x) - u_1(x)u_2(\xi)}{p(\xi)W(u_1, u_2)(\xi)} & \xi \leq x. \end{cases} \]

Recall that the Heaviside function is defined by

\[ H(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases} \]

Let \( u_\xi(x) \) denote the solution of

\[ L(u_\xi(x)) = 0 \]

\[ u_\xi(\xi) = 0 \]

\[ u_\xi'(\xi) = \frac{1}{p(\xi)}. \]

Thus we see that the Green’s function for the initial value problem satisfies

\[ G(x, \xi) = H(x - \xi)u_\xi(x). \]

Such a Green’s function is often referred to as the causal fundamental solution.

For more general boundary conditions, we might seek \( G(x, \xi) \) in the form

\[ G(x, \xi) = H(x - \xi)u_\xi + Au_1(x) + Bu_2(x) \]

where \( u_1, u_2 \) are linearly independent solutions of \( L = 0 \).
Example 4.7. Construct the Green’s function for

\[ u'' = 0, \quad 0 < x < 1 \]
\[ u(0) + u(1) = 0 \]
\[ u'(0) + u'(1) = 0 \]

Seek

\[ G(x, \xi) = H(x - \xi)u_\xi(x) + Ax + B \equiv E(x, \xi) + Ax + B \]

where

\[ u''_\xi = 0, \quad x > \xi > 0 \]
\[ u_\xi(\xi^+) = 0, \quad u_\xi'(\xi^+) = 1. \]

Then

\[ E(x, \xi) = \begin{cases} 
0 & 0 \leq x \leq \xi \\
 x - \xi & x \leq \xi \leq 1 
\end{cases} \]

and

\[ B_1(G) = (E(0, \xi) + B) + (E(1, \xi) + A + B) \]
\[ = 2B + A + (1 + \xi) = 0, \]
\[ B_2(G) = (0 + A) + (1 + A) \]

Solving, one obtains

\[ A = -1/2, \quad B = -1/4 + \xi/2 \]

and hence

\[ G(x, \xi) = \begin{cases} 
\frac{1}{2}x - \frac{1}{4} + \frac{\xi}{2}, & 0 \leq x \leq \xi \\
(x - \xi) - \frac{1}{4} + \frac{\xi}{2} - \frac{\xi}{2}, & x < \xi \leq 1 
\end{cases} \]

or

\[ G(x, \xi) = -\frac{1}{4} + \frac{|x - \xi|}{2} \]
Exercises for Chapter 4

1. If \( q(x) \leq 0 \) on \([a, b]\), then a nontrivial (i.e., non zero) solution of \( y'' + q(x)y = 0 \) on \([a, b]\) can have at most one zero in \([a, b]\).

2. Use Sturm’s Theorem to argue that a nontrivial solution of \( y'' + (1 - e^x)y = 0 \) has at most one zero in \((0, \infty)\) but has infinitely many zeros in \((-\infty, 0)\).

3. Convert the given equation into the following Sturm-Liouville forms

\[
(p(x)y')' + (r(x) + \lambda g(x))y = 0 \quad \text{and} \quad y'' + (q(x) + \lambda)y = 0.
\]

(a) \( y'' + 6y' + \lambda y = 0 \) for \( x \in \mathbb{R} \).
(b) \( x^2y'' + xy' + \lambda y = 0 \) for \( x > 0 \).

4. Let \((\varphi, \lambda)\) be an eigenpair for a Regular Sturm-Liouville Boundary Value Problem (i.e., the equation has the form (4.4.13) with boundary conditions (4.4.14)) on an interval \([a, b]\). Namely, we assume that \( \varphi \) satisfies

\[
\frac{d^2 \varphi}{dx^2} + q(x)\varphi + \lambda \varphi = 0
\]

and the boundary conditions

\[
B_1(\varphi) = \alpha_1 \varphi(a) + \alpha_2 \varphi'(a) = 0, \quad B_2(\varphi) = \beta_1 \varphi(b) + \beta_2 \varphi'(b) = 0,
\]

(a) Show that

\[
\lambda \| \varphi \|^2 = \| \varphi' \|^2 - \int_a^b q(x)|\varphi(x)|^2 dx - \left[ \varphi(x)\varphi'(x) \right]_{x=a}^{x=b} \quad (*)
\]

where \( \| \varphi \|^2 = \int_a^b |\varphi(x)|^2 dx \) is the norm in \( L^2(a, b) \).

(b) Use (*) to show that if \( q(x) \leq 0 \) and \( \alpha_2 = \beta_2 = 0 \) in (4.4.14), then \( \lambda \) is positive.

(c) Show that if \( q(x) \leq 0 \) and \( \alpha_1 = \beta_1 = 0 \) in (4.4.14), then \( \lambda \) is non-negative.

(d) Show that if \( q(x) \leq 0 \) and \( \alpha_1/\alpha_2 < 0 \) and \( \beta_1/\beta_2 > 0 \) in (4.4.14), then \( \lambda \) is non-negative.

5. Find the eigenvalues and eigenfunctions of \( u'' + \lambda u = 0 \) with the boundary conditions:

(a) \( u(0) = u(1) = 0 \),
(b) \( u'(0) = u'(1) = 0 \),
(c) \( u(0) = 0, u(1) - u'(1) = 0 \).

6. Find the Green’s function for

\[
y'' - 4y = f, \quad y'(0) = 0, \quad y'(\pi) = 0.
\]
7. Find the Green’s function for

\[ y'' - \gamma^2 y = f, \quad y'(0) = 0, \quad y(1) = 0, \quad \gamma > 0. \]

8. Find the solution of the heat problem in terms of a Fourier Series.

\[ \frac{\partial u}{\partial t} = 5 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1, \quad t > 0, \]

\[ u(0, t) = u(1, t) = 0 \]

\[ u(x, 0) = (1 - x)x^2. \]

9. Find the solution of the heat problem in terms of a Fourier Series.

\[ \frac{\partial^2 u}{\partial t^2} = 4 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi, \quad t > 0, \]

\[ u(0, t) = u(\pi, t) = 0 \]

\[ u(x, 0) = (\pi - x)x^2, \quad \frac{\partial u}{\partial t}(x, 0) = 0. \]

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