Prelim Exercise Study Set

PART I

1. Consider the following linear systems

$$x'(t) = A(t)x(t), \tag{LH}$$

$$x'(t) = A(t)x(t) + f(t), \quad x(t_0) = x_0, \tag{LNH}$$

where $x(t), f(t) \in \mathbb{R}^n$, $A(t)$ is a real $n \times n$ matrix, and $A(t), f(t)$ are continuous on an open interval I that contains t_0.

   (a) Define what is meant by a fundamental matrix of (LH), explain why it exists, and derive a formula for a solution of the initial value problem (LNH).

   (b) Prove that the unique solution of (LNH) exists on the whole interval I, whether it be finite or infinite.

2. In (LH) suppose that $A(t)$ is constant and given by

$$A = \begin{bmatrix} \alpha_1 & \beta_1 & 0 & 0 \\ -\gamma_1 & \alpha_1 & 0 & 0 \\ 0 & 0 & \alpha_2 & \beta_2 \\ 0 & 0 & -\gamma_2 & \alpha_2 \end{bmatrix}.$$

   (a) If $\alpha_1 = \alpha_2 = 0$ and $\gamma_1 = \beta_1$ and $\gamma_2 = \beta_2$, show that all solutions of (LH) are periodic if β_1/β_2 is rational.

   (b) Suppose that $\gamma_1 = \gamma_2 = \alpha_1 = \beta_2 = 0$ and $\beta_1 = \alpha_2 = 1$. True or False: All solutions of (LH) are unbounded. Explain your answer by proof or counter-example.

3. Rewrite the nth order linear equation

$$z^{(n)}(t) + a_1(t)z^{(n-1)}(t) + \cdots + a_{n-1}(t)z'(t) + a_1(t)z = b(t) \tag{IVP}$$

$$z(0) = z_1,\ z'(0) = z_2,\ \cdots, z^{(n-1)}(0) = z_n$$

(with $a_j(t) \in C(\mathbb{R})$) in the form

$$y'(t) = A(t)y(t) + B(t), \quad y(0) = y_0$$

   (a) Use this form to explain why the (IVP) has a a globally defined solution for all initial data.

   (b) Define a fundamental matrix for $\dot{y} = Ay$ and explain why a fundamental matrix exists.

   (c) Derive Abel's formula for (IVP) If $y_1, \ldots y_n$ are solutions of (LH), and $t_0 \in (a, b)$, then

$$W(t) = W(t_0) \exp\left[-\int_{t_0}^{t} \alpha_1(s)ds\right].$$

   where W is the Wronskian of $\{y_1, \cdots y_n\}$.

4. Establish the following two versions of Gronwall's Inequality
   (a) Let $f_1(t)$, $f_2(t)$, $p(t)$ be continuous on [a,b] and $p \geq 0$. If

   $$f_1(t) \leq f_2(t) + \int_a^t p(s) f_1(s)\, ds, \quad t \in [a,b],$$

   then

   $$f_1(t) \leq f_2(t) + \int_a^t p(s) f_2(s) \exp[\int_s^t p(u) du]\, ds.$$

   (b) Give a simpler derivation in the special case $p(t) = k$ and $f_2(t) = \delta$ are constant, i.e., assume that

   $$f_1(t) \leq \delta + k \int_a^t f_1(s)\, ds, \quad t \in [a,b].$$

   Show that in this case $f_1(t) \leq \delta e^{k|x-a|}$.

5. Do parts a) and b)
   (a) Suppose $x(t) = 0$ is a solution of $\dot{x} = f(t,x)$ where $x \in \mathbb{R}^n$ and $f(t,x)$ is a smooth function. Define what it means to say that $x(t) = 0$ is a stable solution. Define what it means to say that $x(t) = 0$ is asymptotically stable.

   (b) Consider $\ddot{x} + \mu\dot{x} + x + x^3 = 0$ where $\mu > 0$. State an appropriate theorem and use it to show that $x(t) = 0$ is asymptotically stable.

6. Stability of nonlinear systems:
   (a) Let A be an $n \times n$ real constant matrix with $\Re(\sigma(A)) < 0$. Let g be a $\mathbb{R}^n$ valued function which is $C^1(\mathbb{R}^n \times [0,\infty)$. We assume in addition that $g(t,0) = 0$ for all t. State a theorem that will prove that the origin is an asymptotically stable fixed point for the nonlinear system

   $$x'(t) = Ax(t) + g(t, x(t)).$$

   (b) If $c_1, c_2 > 0$ show that the equilibrium solution is asymptotically stable for

   $$\begin{aligned} \dot{x_1} &= (x_1 - c_2 x_2)(x_1^2 + x_2^2 - 1) \\ \dot{x_2} &= (c_1 x_1 + x_2)(x_1^2 + x_2^2 - 1) \end{aligned}$$

   (c) For a system $x'(t) = f(x(t))$ with $f \in C^1(\mathbb{R}^n \times [0,\infty))$ define a Lyapunov function and state carefully the main Lyapunov stability theorem.

   (d) Construct a Lyapunov function to show that the origin is an asymptotically stabile equilibrium for

   $$\begin{aligned} x' &= -y - x^3 \\ y' &= x - y^3 \end{aligned}$$

7. Do all parts
   (a) Show that $x = 0$ is unstable stable for

$$\begin{aligned} \dot{x_1} &= x_1 + x_2 \\ \dot{x_2} &= x_1 - x_2 + x_1 x_2 \end{aligned}$$

   (b) If $f(x, y) \geq 0$ show that $x = 0$ is stable for

$$\ddot{x} + f(x, \dot{x})\dot{x} + \omega^2 x = 0.$$

   (c) If $g(0) = 0$ and $xg(x) > 0$ show that $x(t) = 0$ is a stable equilibrium for

$$x''(t) + g(x(t)) = 0.$$

8. Consider the system

$$Y'(t) = AY(t) + B(t) \qquad (*)$$

where A is an $n \times n$ constant matrix and $B(t)$ is a continuous $n-$dimensional vector that satisfies the condition $\int_0^\infty |B(t)| dt < \infty$.

   (a) Suppose that the eigenvalues of A have negative real parts. Show that all solutions of (*) are bounded, i.e., there is a constant M so that $|Y(t)| \leq M$ for $t \geq 0$.
   (b) Suppose we only assume that $B(t)$ is bounded. Is (a) true?
   (c) Find a condition on α so that $x = 0$ is stable for

$$\ddot{x} + \left[1 + \frac{t}{(1+t)^\alpha}\right] x = 0$$

9. Show that a solution to the equation

$$y''(x) + x^2 y(x) = 0, \quad y(0) = 0,$$

has a zero in the interval $(0, 2)$. HINT: Note that the equation

$$z'' + \lambda^2 z = 0.$$

has a solution that satisfies $z(0) = 0$ and $z(\pi/\lambda) = 0$. Then consider

$$\int_0^{\pi/\lambda} (x^2 - \lambda^2) \sin^2(\lambda x)\, dx.$$

10. Consider the Sturm-Liouville problem,

$$(xy'(x))' + \frac{\lambda}{x} y(x) = 0, \;\; y(1) = y(e) = 0.$$

   (a) Show that the eigenvalues and eigenfunctions are given by

$$\lambda_n = n^2\pi^2, \;\; y_n(x) = \sin(n\pi \ln x), \quad n = 1, 2, 3, \cdots$$

(b) Construct the Green's function for this problem if $\lambda = 1$.

(c) Consider the nonhomogeneous problem

$$(xy'(x))' + \frac{\pi^2}{x} y(x) = \sin(n\pi \ln x), \;\; y(1) = y(e) = 0.$$

For which integers n is this problem solvable?

11. Consider the Bessel equation

$$y''(x) + \left(1 + \frac{1 - 4p^2}{4x^2}\right) y(x) = 0$$

(a) If $0 \leq p < 1/2$, show that every nontrivial solution of the Bessel equation has at least one zero in every interval of length π.
(b) If $p = 1/2$, show the zeros of every solution are separated by an interval of length π.
(c) If $p > 1/2$, show that every solution can have at most one zero in any interval of length π.

12. Find the eigenvalues and eigenfunctions of $u'' + \lambda u = 0$ with the boundary conditions:

(a) $u(0) = u(\ell) = 0$,
(b) $u'(0) = u'(\ell) = 0$,
(c) $u(0) = 0, u(1) - u'(\ell) = 0$.

13. Find the Green's function for

$$y'' - \gamma^2 y = 0, \;\; y'(0) = 0, \;\; y(\ell) = 0, \;\; \gamma > 0.$$

14. Find the eigenvalues and eigenfunctions for the following boundary value problem and then construct the Green's function.

$$\frac{d}{dx}\left(x\frac{dy}{dx}\right) + \frac{\lambda}{x} y = 0, \;\; 1 < x < e$$

$$u(1) = 0, \; u(e) = 0.$$

PART II

1. Show that

$$(1 + x^2)u_{xx} + (1 + y^2)u_{yy} + xu_x + yu_y = 0$$

is elliptic and find the canonical form of the equation.

2. Solve

$$u_x + u_y = u^2$$

with the initial condition $u(x, 0) = h(x)$.

3. For the problem

$$u^2 u_x + u_y = 0, \quad x \in \mathbb{R}, \quad y > 0$$
$$u(x,0) = x$$

derive the solution

$$u(x,y) = (\sqrt{1+4xy} - 1)/2y$$

valid for $y \neq 0, \quad 1 + 4xy > 0$. Verify that $u(x,y)$ satisfies the initial condition. When do shocks develop?

4. Consider the differential equation

$$u_{xx} - 5u_{xy} + 6u_{yy} = 0.$$

   (a) Classify the equation and reduce it to canonical form.

   (b) Integrate the canonical form of the equation to obtain the solution

   $$u(x,y) = f(y+3x) + g(y+2x)$$

   where f, g are arbitrary C^2 functions.

5. Consider the differential equation

$$uu_x + u_y = 1$$

with the initial condition $u = s/2$ on the curve $x = y = s, \ 0 < s < 1$. Verify that this problem has a unique solution in a neighborhood of the initial curve and then find it.

6. Let $u(x,t)$ solve the initial value problem

$$\Delta u = u_{tt}, \quad x \in \mathbb{R}^3, \ t > 0$$
$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x)$$

where u_0, u_1 are infinitely differentiable and vanish outside some ball $B(0,R)$. Define the energy of u in the ball $|x| \leq \delta$ at time t by

$$E_\delta[u(t)] = \frac{1}{2} \int_{|x| \leq \delta} \left\{ |\nabla u(x,t)|^2 + u_t^2(x,t) \right\} \, dx$$

and the energy of u in $\mathbb{R}^3$ by

$$E_\infty[u(t)] = (1/2) \int_{R^3} \left\{ |\nabla u(x,t)|^2 + u_t^2(x,t) \right\} \, dx$$

   (a) Show that the energy of u contained in the whole space $\mathbb{R}^3$ is constant, i.e.,

   $$E_\infty[u(t)] = E_\infty[u(0)], \quad t > 0.$$

   (b) Show that

   $$\lim_{t \to \infty} E_\delta[u(t)] = 0.$$

7. Solve the problem

$$\begin{gathered}\Delta u = u_{tt}, \quad x \in \mathbb{R}^3, x_3 > 0,\ t > 0\\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad x \in \mathbb{R}^3, x_3 \geq 0\\ u_{x_3}(x_1, x_2, 0, t) = 0, \quad -\infty < x_1, x_2 < \infty,\ t \geq 0,\end{gathered}$$

   by introducing the appropriate imaginary data in the lower half space $x_3 < 0$.

8. Solve

$$\begin{aligned} u_{tt} &= u_{xx} + f(x,t), \quad x > 0,\ t > 0\\ u(x,0) &= u_t(x,0) = 0 \quad x > 0\\ u(0,t) &= h(t), \quad t \geq 0\end{aligned}$$

9. State Duhamel's Principle and use it to solve

$$\begin{aligned} u_{tt} &= u_{xx} + x^2, \quad x \in \mathbb{R},\ t > 0\\ u(x,0) &= x, \quad u_t(x,0) = 0\end{aligned}$$

10. Suppose $u(x,t)$ is a solution of

$$\begin{gathered}\Delta u = u_{tt}, \quad x \in \mathbb{R}^n,\ t > 0\\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) \quad x \in \mathbb{R}^n\end{gathered}$$

    where u_0, u_1 are smooth functions with compact support.

    (a) If $n = 3$, show that for each x there exists a time $T(x)$ such that if $t > T(x)$, $u(x,t) = 0$.

    (b) If $n = 2$, show that for each x

$$\lim_{t \to \infty} u(x,t) = 0.$$

    (c) If $n = 1$, show that for each x there exists a time $T(x)$ such that if $t > T(x)$, then $u(x,t)$ is a constant.

11. Solve

$$\begin{aligned} u_{tt} &= u_{xx} + \alpha u_t + \alpha u_x, \quad x \in \mathbb{R},\ t > 0\\ u(x,0) &= u_0(x), \quad u_t(x,0) = u_1(x).\end{aligned}$$

12. Do both parts

    (a) Let Ω be a piecewise smooth bounded domain in $\mathbb{R}^n$ and suppose $u(x,t)$ is a smooth solution of the initial boundary value problems

$$\begin{array}{cl} \Delta u = u_{tt}, & x \in \Omega,\ t > 0\\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in \Omega\\ u(x) = 0, & x \in S_1\\ \dfrac{\partial u}{\partial n} = 0, & x \in S_2, \quad \partial\Omega = S_1 \cup S_2\end{array}$$

    If

$$\mathcal{E}(t) = \int_\Omega \left[|\nabla u(x,t)|^2 + u_t^2(x,t) \right] dx$$

    show that $\mathcal{E}(t)$ is constant.

(b) Prove that there is at most one smooth solution of the initial boundary value problem

$$\begin{array}{rl} u_{tt} = u_{xx} + f(x,t), & 0 < x < \ell,\ t > 0 \\ u(x,0) = u_0(x),\ \ u_t(x,0) = u_1(x), & 0 \leq x \leq \ell \\ u(0,t) = g(t),\ \ u_x(\ell,t) = h(t), & t \geq 0 \end{array}$$

13. Solve the nonhomogeneous initial boundary value problem

$$u_{tt} = u_{xx} + x, \quad 0 < x < 1,\ t > 0$$

$$u(x,0) = \sin(\pi x/2), \qquad u_t(x,0) = 0 \ \ 0 \leq x \leq 1$$

$$u(0,t) = 0, \qquad u_x(1,t) = \sin(t) t \geq 0$$

14. Recall that a fundamental solution of $\Delta u = 0$ is given by

$$k(x,y) = \begin{cases} \dfrac{1}{2\pi} \ln \dfrac{1}{|x-y|}, & n = 2 \\[2ex] \dfrac{1}{4\pi} \dfrac{1}{|x-y|}, & n = 3. \end{cases}$$

a) For $n = 2$ (or $n = 3$), derive the following representation formula

$$u(x) = \int_{\partial\Omega} k(x,y) \frac{\partial u}{\partial n}(y) - u(y) \frac{\partial}{\partial n} k(x,y) d\sigma_y - \int_{\Omega} k(x,y) \Delta u \, dy \tag{*}$$

where Ω and $u(x)$ satisfy appropriate regularity conditions.

b) Define what is meant by a Green's function for the boundary value problem (BVP)

$$\Delta u = 0, \quad x \in \Omega,$$

$$u(x) = f(x), \quad x \in \partial\Omega.$$

Use $(*)$ to express the solution of (BVP) in terms of the Green's function.

c) Construct the Green's function for (BVP) when Ω is the upper half-plane (or the upper half-space) and derive a formula for the solution of (BVP).

15. Consider the boundary value problem

$$\Delta u + c(x) u = h(x), \quad x \in \Omega$$

$$\frac{\partial u}{\partial n} = f(x), \quad x \in \partial\Omega,$$

where Ω is a normal domain in R^n, $f \in C(\partial\Omega)$ and $c, h \in C(\bar{\Omega})$. Prove that this problem has at most one solution in $C^2(\bar{\Omega})$ if $c(x) < 0$. Show that any two solutions in $C^2(\bar{\Omega})$ differ by a constant if $c(x) \equiv 0$

16. Let $\Omega \subset B(0,R) \subset \mathbb{R}^2$

a) Let $\Delta u = -F$ in Ω and suppose that $F \leq 0$ in Ω. If in addition $u \in C(\overline{\Omega})$, then

$$\max_{x \in \Omega} u(x) \leq \max_{x \in \partial\Omega} u(x).$$

b) Consider the nonhomogeneous Dirichlet Problem

$$\begin{aligned} &\Delta u = -F \quad \text{in } \ \Omega \subset B(0, R) \\ &u = f \quad \text{in } \ \partial\Omega. \end{aligned}$$

Show that

$$|u(x,y)| \leq \max_{(x,y) \in \partial\Omega} |f(x,y)| + \frac{1}{4} R^2 \max_{(x,y) \in \Omega} |F(x,y)|.$$

17. Prove that if u is harmonic in a bounded domain $\Omega \subset \mathbb{R}^2$ and is $C^2(\overline{\Omega})$, then $|\nabla u|^2$ attains its maximum on $\partial\Omega$.

18. Uniqess in unbounded domains

    (a) Show that the exterior DP

    $$\Delta u(x) = 0, x \in \Omega,$$
    $$u = 1$$

    has infinitely many solutions.

    (b) If for $n > 2$ and we impose the condition that $u(x) \to 0$ as $|x| \to \infty$, then the problem has a unique solution.

19. Show that the exterior DP (i.e., $\Omega^c = \mathbb{C} \backslash \Omega$ is a nonempty bounded domain.)

$$\begin{aligned} &u_{xx} + u_{yy} = f \quad \in \Omega \ (\textit{unbounded}), \\ &u = g \ \text{ on } \ \partial\Omega, \\ &|u(x,y)| \leq A \ \ (x,y) \in \Omega, \end{aligned}$$

has at most one solution.

20. Consider the nonhomogeneous heat equation

$$u_t = u_{xx} + f(x,t), \quad x \in \mathbb{R}, \ t > 0$$

$$u(x,0) = 0, \quad x \in \mathbb{R}.$$

Apply Duhamel's principle to find the following formula for the solution of this problem:

$$u(x,t) = \int_0^t \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi(t-\tau)}} e^{\frac{-(x-\xi)^2}{4(t-\tau)}} f(\xi, \tau) \, d\xi d\tau.$$

21. Recall that the solution to the initial value heat problem

$$u_t = u_{xx}, \quad x \in \mathbb{R}, \quad t > 0$$

$$u(x,0) = f(x)$$

is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4t}} f(y) dy.$$

a) Prove that the solution depends continuously on the data in the sense that if

$$|f(x) - \tilde{f}(x)| < \epsilon, \quad -\infty < x < \infty,$$

then the corresponding solutions satisfy

$$|u(x,t) - \tilde{u}(x,t)| < \epsilon, \quad -\infty < x < \infty, t > 0.$$

b) Assume that f(x) is continuous and bounded. Show that

$$\lim_{t \to 0^+} u(x,t) = f(x)$$

22. The heat equation in the semi-infinite rod with its end kept at zero temperature or being insulated leads to the initial-boundary value problems

$$u_t = u_{xx}, \quad 0 < x < \infty, \ t > 0$$

$$u(x,0) = \phi(x), \quad 0 \leq x < \infty$$

$$u(0,t) = 0, \quad t \geq 0 \tag{1}$$

or

$$u_x(0,t) = 0 \quad t \geq 0. \tag{2}$$

For the boundary condition (1) show that

$$u(x,t) = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{1}{\sqrt{t}} \left[e^{\frac{-(x-\xi)^2}{4t}} - e^{\frac{-(x+\xi)^2}{4t}} \right] \phi(\xi)\, d\xi.$$

For the boundary condition (2) show that

$$u(x,t) = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{1}{\sqrt{t}} \left[e^{\frac{-(x-\xi)^2}{4t}} + e^{\frac{-(x+\xi)^2}{4t}} \right] \phi(\xi)\, d\xi.$$

23. Consider the initial value problem (IVP) for the temperature in an infinitely long rod,

$$u_t = u_{xx}, \quad x \in \mathbb{R}, \ t > 0$$

$$u(x,0) = \begin{cases} T_0 & x \geq 0 \\ 0 & x < 0 \end{cases}.$$

a) Show that the solution of (IVP) is given by

$$u(x,t) = \frac{T_0}{2} + \frac{T_0}{\sqrt{\pi}} \int_0^{x/\sqrt{4t}} e^{-\alpha^2}\, d\alpha$$

b) Note that the solution is positive and infinitely differentiable for $t > 0$. What is the steady state value of $u(x,t)$?

24. Consider the initial boundary value problem,

$$\begin{aligned} u_t = u_{xx} + F(x,t), &\quad 0 < x < l,\ t > 0 \\ u(x,0) = 0, &\quad 0 \le x \le l \\ u(0,t) = 0, \quad u(l,t) = 0, &\quad t \ge 0. \end{aligned}$$

Use Duhamel's principle and a formal series solution to obtain the following formula for the solution of the nonhomogeneous problem,

$$u(x,t) = \sum_{k=1}^{\infty} \left\{ \int_0^t F_k(s) e^{-\frac{k^2\pi^2(t-s)}{l^2}}\, ds \right\} \sin \frac{k\pi x}{l}$$

where

$$F_k(s) = \frac{2}{l} \int_0^l F(x,s) \sin \frac{k\pi x}{l}\, dx.$$

If $F(x,t) = F(t)\sin(\pi x/l)$ show that the solution is

$$u(x,t) = \left(\int_0^t F(s) e^{\frac{\pi^2 s}{l^2}}\, ds \right) \sin \frac{\pi x}{l} e^{-\frac{\pi^2 t}{l^2}}$$

25. More heat equation on a finite interval

a) Separate variables to construct a series solution of

$$\begin{aligned} u_t = u_{xx}, &\quad 0 < x < \pi,\ t > 0 \\ u(x,0) = x, &\quad 0 \le x < \pi \\ u_x(0,t) = 0 = u_x(\pi,t), &\quad t \ge 0 \end{aligned}$$

b) Carefully justify that the series solution satisfies the boundary conditions and the initial condition and show that for each $t > 0$ the function $u(x,t)$ defined by this series represents a C^∞ function in x that satisfies the heat equation.