Chapter 9

Parabolic Equations

9.1 Introduction

If $u(x, t)$ represents the temperature in a body $\Omega \subset \mathbb{R}^n$ at a point $x \in \Omega$ at time t, then the propagation (or diffusion) of heat is (approximately) described by the heat equation

$$u_t = \Delta u.$$

The heat equation is the usual example of a parabolic equation that one finds in most books on partial differential equations. In what follows we will attempt to describe some of the basic ingredients in a classical disscussion of the heat equation. At the same time we try to provide the student with a brief introduction to some important tools used in PDEs which have not been discussed up to this point. One such tool that could have used all along is the Fourier transform. One problem is that to use this tool most effectively the student should have had a course in real analysis. Since this is not required for the present class it presents a situation in which we must strike a delicate balance between what we would like to do and what we can really do rigorously. At some point I will present results without proof and simply give references for details.

Following [3] we first give a few caveats concerning the heat equation in physics. As pointed out in [3], the heat equation does not say anything about the microscopic physical processes that accompany heat flow. It merely describes a limiting situation in which the size of the atoms can be considered infinitesimally small. It does not recognize the existence of an absolute zero temperature – indeed, temperatures for the heat equation can even be negative. Nonetheless, the heat equation is a very important an interesting model to study.

9.2 Some Results from Functional Analysis

At this point we digress to discuss several topics from real and functional analysis.

Hölder's inequality is used often in analysis and can be found in almost any book on real or functional analysis, see for example [5, 20]. Since a graduate course in real analysis is not a prerequisite for this course we will present several several results, more or less directly from the book used for real analysis at Texas Tech, "Real Analysis" by Gerald Folland [5]. For this presentation when you see an expression like $\int f(x)\,d\mu(x)$ where we say μ is a measure just replace $d\mu$ by dx and imagine the functions are integrable in the sense you know best. This might be the Riemann integral or even the Lebesgue integral if you had Baby Reals. The strict validity of the results does rely heavily upon the Lebesgue integral but for our purposes we are really only interested in using the results and not in really rigorously proving them – thats what the real analysis is for and you should take it if you have not already. We will be interested in the classical L^p function spaces consisting of functions (not really they are actually equivalence classes of functions that differ on sets of measure zero, but we need not worry about this) f whose pth power is integrable in $\mathbb{R}^n$, i.e.

$$L^p(\mathbb{R}^n) = \{f \,:\, \mathbb{R}^n \to \mathbb{C} \,:\, \|f\|_p < \infty\}$$

where $\|\cdot\|_p$ for $1 \le p < \infty$ is defined by

$$\|f\|_p = \left(\int |f(x)|^p\,dx\right)^{1/p}.$$

The case of $p = 2$ is of special importance and is given the name Hilbert Space. In this case the norm $\|\cdot\|_2$ is induced (just as it is in $\mathbb{C}^n$) from an inner product. Namely we define the inner product in L^2 by

$$\langle f, g\rangle = \int f(x)\overline{g(x)}\,dx$$

and it follows immediately that

$$\|f\|_2^2 = \int |f(x)|^2\,dx = \int f(x)\overline{f(x)}\,dx = \langle f, f\rangle.$$

Theorem 9.2.1 (Hölder's Inequality). *Suppose that $1 < p < \infty$ and let q satisfy $p^{-1} + q^{-1} = 1$ (in this case we say that p and q are conjugate exponents). If f and g are measureable functions on a measure space (X, μ) then*

$$\|fg\|_1 \le \|f\|_p\|g\|_q.$$

In particular, if $f \in L^p(X,\mu)$ and $g \in L^q(X,\mu)$, then $fg \in L^1(X,\mu)$.

The special case $p = q = 2$ is called the Cauchy-Schwartz Inequality.

One result which has far reaching consequences is the following which is taken from [3].

Theorem 9.2.2 (Generalized Young' Inequality). *Let (S, μ) be a measure space $1 \leq p \leq \infty$ and $C > 0$. Suppose K is a measureable function on $S \times S$ such that*

$$\int_S |K(x,y)| d\mu(y) \leq C, \ \forall \ x \in S$$

and

$$\int_S |K(x,y)| d\mu(x) \leq C, \ \forall \ y \in S$$

and suppose that $f \in L^p(S)$. Then

$$Tf(x) = \int_S K(x,y) f(y)\, dy$$

is defined almost everywhere and $Tf \in L^p(S)$ with

$$\|Tf\|_p \leq C \|f\|_p.$$

An indispensible tool is the convolution integral. For f and g in $L^1_{loc}(\mathbb{R}^n)$ we define formally the convolution of f and g by

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y) g(y)\, dy.$$

Corollary 9.2.1 (Young's inequality). *If $f \in L^1$ and $g \in L^p$ $(1 \leq p \leq \infty)$, then $f * g(x) \in L^p$ and $\|f * g\|_p \leq \|f\|_1 \|g\|_p$.*

Proof. Apply Theorem 9.2.2 with $S = \mathbb{R}^n$ and $K(x,y) = f(x-y)$. $\square$

Remark 9.2.1. If $f \in L^p$ and $g \in L^q$ where $p^{-1} + q^{-1} = 1$, then $(f * g) \in L^\infty$. This follows from Hölder's inequality:

$$\|(f * g)\|_\infty \leq \operatorname{ess\,sup}_{x \in \mathbb{R}^n} \int_{\mathbb{R}^n} |f(x-y)|\, |g(y)|\, dy \leq$$

$$\operatorname{ess\,sup}_{x \in \mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(x-y)|^p\, dy \right)^{1/p} \left(\int_{\mathbb{R}^n} |g(y)|^q\, dy \right)^{1/q}$$

$$= \|f\|_p \|g\|_q.$$

Here

$$\|\varphi\|_\infty = \operatorname{ess\,sup}_{x \in \mathbb{R}^n} |\varphi(x)| = \inf\{\alpha \geq 0 \,:\, \mu(\{x \,:\, |\varphi(x)| \geq \alpha\}) = 0\}.$$

For our course without real analysis, you can assume that φ is say piecewise continuous and

$$\|f\|_\infty = \sup_{x \in \mathbb{R}^n} |f(x)|.$$

Note that from Hölder's inequality, if $f \in L^q$ and $g \in L^p$, $(1/p + 1/q = 1)$, then

$$f * g \in L^\infty$$

and $\|f * g\|_\infty \le \|f\|_q \|g\|_p$.

Definition 9.2.1. *If f is a function on $\mathbb{R}^n$ and $x \in \mathbb{R}^n$ we define $f_x(y) = f(x+y)$*

Lemma 9.2.1. *For $1 \le p < \infty$ and $f \in L^p$,*

$$\lim_{x \to 0} \|f_x - f\|_p = 0.$$

Proof. Since a continuous function on a compact set is uniformly continuous, if g is a continuous, compactly supported function, then g is also uniformly continuous. Thus for such a g, given $\epsilon > 0$ there exists an δ so that $|g(x+y) - g(y)| < \epsilon$ for all y and provided $|x| \le \delta$. If we fix a compact set containing the support of g_x for all $|x| \le 1$, then we can assume that g and g_x are supported on a common compact set for $|x| \le 1$. Thus we have $\|g_x - g\|_p \to 0$ as $x \to 0$. Now for a given $f \in L^p$ and $\epsilon > 0$, choose a continuous g with compact support such that $\|f - g\|_p \le \epsilon/3$. Then we also have $\|f_x - g_x\|_p \le \epsilon/3$ (by a simple change of variables in the integral), so that for all $|x| \le 1$

$$\|f_x - f\|_p \le \|f_x - g_x\|_p + \|g_x - g\|_p + \|g - f\|_p \le \|g_x - g\|_p + 2\epsilon/3.$$

Now for x sufficiently small we have $\|g_x - g\|_p \le \epsilon/3$ and we have

$$\|f_x - f\|_p \le \epsilon.$$

□

Remark 9.2.2. The result is not true for $p = \infty$. In fact the condition that $\|f_x - f\|_\infty \to 0$ as $x \to 0$ means that f agrees (a.e.) with a uniformly continuous function.

Remark 9.2.3. Minkowski's inequality says that the L^p norm of the sum of a finite number of functions is less than or equal to the sum of the L^p norms of the functions. This result is also true in an integral form (See J. Folland's book on real analysis, [5] page 186). Namely, if (X, μ) and (Y, ν) are σ-finite measure spaces, $f(x,y) \ge 0$ and $1 \le p < \infty$, then we have

$$\left[\int \left| \int f(x,y)\, d\nu(y) \right|^p d\mu(x) \right]^{1/p} \le \int \left[\int f(x,y)^p \, d\mu(x) \right]^{1/p} d\nu(y). \qquad (*)$$

Theorem 9.2.3. *Let $\phi \in L^1(\mathbb{R}^n)$, $\epsilon > 0$, $\phi_\epsilon(x) = \epsilon^{-n} \phi\left(\frac{x}{\epsilon}\right)$ and $\int_{\mathbb{R}^n} \phi(x)\, dx = a$. If $f \in L^p$, $1 \le p < \infty$, then*

$$\|f * \phi_\epsilon - af\|_p \to 0, \ \epsilon \to 0$$

If f is bounded and uniformly continuous, then

$$\|f * \phi_\epsilon - af\|_\infty \to 0, \ \epsilon \to 0$$

Proof. By the change of variables $x \to \epsilon x$ we see that $\displaystyle\int \varphi_\epsilon(x)\,dx = a$ for all $\epsilon > 0$. Therefore we have

$$f * \varphi_\epsilon(x) - af(x) = \int [f(x-y) - f(x)]\,\varphi_\epsilon(y)\,dy = \int [f(x-\epsilon y) - f(x)]\,\varphi(y)\,dy.$$

If $f \in L^p$ and $p < \infty$ then we can apply the Minkowski inequality for integrals $(*)$ given in Remark 9.2.3, to conclude that

$$\|f * \varphi_\epsilon - af\|_p \leq \int \|f_{-\epsilon y} - f\|_p\,|\varphi(y)|\,dy.$$

But $\|f_{-\epsilon y} - f\|_p$ is bounded by $2\|f\|_p$ and tends to zero as $\epsilon \to 0$ for each y, by Lemma 9.2.1. Thus we can apply the Lebesgue Dominated Convergence Theorem to get

$$\lim_{\epsilon\to 0} \|f * \varphi_\epsilon - af\|_p \leq \lim_{\epsilon\to 0} \int \|f_{-\epsilon y} - f\|_p\,|\varphi(y)|\,dy = \int \lim_{\epsilon\to 0} \|f_{-\epsilon y} - f\|_p\,|\varphi(y)|\,dy = 0.$$

If $p = \infty$ and $f \in L^\infty$ is *uniformly continuous* on a set V. Given $\delta > 0$, choose a compact set W so that $\displaystyle\int_{\mathbb{R}^n\setminus W} |\varphi| < \delta$. Then

$$\sup_{x\in V} |(f * \varphi_\epsilon)(x) - af(x)| \leq \sup_{x\in V, y\in W} |f(x-\epsilon y) - f(x)| \int_W |\varphi| + 2\|f\|_\infty \delta.$$

The first term on the right tends to zero as $\epsilon \to 0$, and δ is arbitrary, so $f * \varphi_\epsilon$ tends uniformly to (af) on V. □

Definition 9.2.2. *If $\phi \in L^1$ and $\int \phi = 1$, then the functions ϕ_ϵ are called an approximation to the Identity.*

Definition 9.2.3. *The Schwartz class $\mathcal{S}$ of rapidly decreasing functions consists on all C^∞ function on $\mathbb{R}^n$ which, together with there derivatives, go to zero at infinity faster than any power of $|x|$. More precisely, for any nonnegative integer N and any multi-index α we define*

$$\|f\|_{(N,\alpha)} = \sup_{x\in\mathbb{R}^n} (1+|x|)^N\,|\partial^\alpha f(x)|$$

and

$$\mathcal{S} = \{f \in C^\infty(\mathbb{R}^n) \,:\, \|f\|_{(N,\alpha)} < \infty, \quad \text{for all } N, \alpha\}.$$

Theorem 9.2.4. *If $f \in L^1$ $(1 \leq p \leq \infty)$ and $\phi \in \mathcal{S}$ then $f * \phi \in C^\infty$ and $\partial^\alpha(f * \phi) = f * (\partial^\alpha \phi)$, for all α.*

Proof. Let $\varphi \in \mathcal{S}$, then for any bounded set $V \subset \mathbb{R}^n$ we have

$$\sup_{x\in V} \left|\partial^\alpha \varphi(x-y)\right| \le C_{\alpha,V}(1+|y|)^{-(n+1)} \quad \text{for all} \;\; y \in \mathbb{R}^n.$$

The function $(1+|y|)^{-(n+1)}$ is in L^q for every q, so the integral

$$f * \partial^\alpha \varphi(x) = \int f(y)\partial^\alpha \varphi(x-y)\,dy$$

converges absolutely and uniformly on bounded subsets of $\mathbb{R}^n$. Therefore the derivatives ∂^α can be interchanged with the integration and we conclude that $\partial^\alpha(f*\varphi) = f*\partial^\alpha\varphi$. □

If in the above result we had $\varphi \in C_c^\infty(\mathbb{R}^n)$ then we would only need to assume that $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ for the convolution $f*\varphi$ to be well defined and the same proof as above shows that $f*\varphi \in C^\infty$.

By way of a standard example, we can establish the existence of smooth compactly supported functions: Namely, let

$$f(t) = \begin{cases} \exp\left(\dfrac{1}{1-t^2}\right), & |t|<1 \\ \quad 0 & |t|>1. \end{cases} \tag{9.2.1}$$

Then $f \in C_c^\infty(\mathbb{R})$ and we can have $\psi(x) = f(|x|)$ for $x \in \mathbb{R}^n$. Then

$$\psi \ge 0, \qquad \text{and} \quad 0 < a \equiv \int \psi(x)\,dx < \infty$$

and we define

$$\phi(x) = \frac{\psi(x)}{a} \in C_c^\infty(\mathbb{R}^n) \tag{9.2.2}$$

Another important result is the following generalization of Urysohn's Lemma.

Lemma 9.2.2. *Let $V \subset \mathbb{R}^n$ be compact and $\Omega \subset \mathbb{R}^n$ be open such that $V \subset \Omega$. Then there exists $f \in C_0^\infty(\Omega)$ such that $f = 1$ on V and $0 \le f \le 1$ everywhere.*

One of the most important concepts in PDEs is the classical partition of unity due to J.P. Dieudonne. The above result is often used in the construction of a partition of unity.

Once again following [3], we define the Fourier transform and list many of its properties.

Definition 9.2.4. *For $f \in L^1(\mathbb{R}^n)$ we define the Fourier transform of f by*

$$\mathcal{F}(f)(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi} f(x)\,dx \tag{9.2.3}$$

Clearly $\widehat{f}(\xi)$ is well defined for every ξ.

More precisely, we have

Theorem 9.2.5. *For $f \in L^1$, $\widehat{f} \in L^\infty$ and*

$$\|\widehat{f}\|_\infty \le \|f\|_1.$$

Proof. This is obvious since

$$|\widehat{f}(\xi)| = \left|\int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi} f(x)\,dx\right| \le \int_{\mathbb{R}^n} |f(x)|\,dx = \|f\|_1.$$

□

Theorem 9.2.6. *1. If $f, g \in L^1$, then $\mathcal{F}(f * g) = \mathcal{F}(f)\mathcal{F}(g)$.*

2. If $f \in \mathcal{S}$, then $\partial^\beta \widehat{f} = \widehat{g}$ where $g(x) = (-2\pi i x)^\beta f(x)$.

3. If $f \in \mathcal{S}$, then $\mathcal{F}(\partial^\alpha f)(\xi) = (2\pi i\xi)^\alpha \widehat{f}(\xi)$.

4. If $f \in \mathcal{S}$, then $\widehat{f} \in \mathcal{S}$.

Proof. 1. First we note that by the Generalized Young's inequaltiy, since $f, g \in L^1$, then $(f * g) \in L^1$. Now the result follows by applying Fubini's theorem (cf, [3]) in order to justify an interchange of integrations. Namely we have

$$\begin{aligned}
\mathcal{F}(f * g)(\xi) &= \iint e^{-2\pi i x\cdot\xi}\, f(x-y)g(y)\,dxdy \\
&= \iint e^{-2\pi i (x-y)\cdot\xi}\, f(x-y)\, e^{-2\pi i y\cdot\xi} g(y)\,dxdy \\
&= \int \left(\int e^{-2\pi i (x-y)\cdot\xi}\, f(x-y)\,dx\right) e^{-2\pi i y\cdot\xi} g(y)\,dy, \quad \text{let } w = (x-y), \\
&= \int \left(\int e^{-2\pi i \xi\cdot w}\, f(w)\,dw\right) e^{-2\pi i y\cdot\xi} g(y)\,dy \\
&= \widehat{f}(\xi) \int e^{-2\pi i y\cdot\xi} g(y)\,dy \\
&= \widehat{f}(\xi)\,\widehat{g}(\xi).
\end{aligned}$$

2. Just differentiate under the integral sign.

$$\frac{\partial}{\partial \xi_j}\widehat{f}(\xi) = \int \frac{\partial}{\partial \xi_j} e^{-2\pi i \xi\cdot x}\, f(x)\,dx = \int (-2\pi i x_j) e^{-2\pi i \xi\cdot x}\, f(x)\,dx = \mathcal{F}((-2\pi i x_j) f).$$

3. Write out the integral and integrate by parts.

$$\begin{aligned}\mathcal{F}(\partial_{x_j} f)(\xi) &= \int e^{-2\pi i\xi\cdot x}\,\partial_{x_j} f(x)\,dx \\ &= (2\pi i\xi_j)\int e^{-2\pi i\xi\cdot x}\,f(x)\,dx \\ &= (2\pi i\xi_j)\mathcal{F}(f)(\xi).\end{aligned}$$

4. By parts 2. and 3.

$$\partial^\beta(\xi^\alpha \widehat{f}(\xi)) = (-1)^{|\beta|}(2\pi i)^{|\beta|-|\alpha|}\mathcal{F}\left[x^\beta\partial^\alpha f\right],$$

so $\partial^\beta(\xi^\alpha\widehat{f}(\xi)$ is bounded for all α and β. Then using the product rule for derivatives and mathematical induction β it can be proven that $\xi^\alpha\partial^\beta\widehat{f}$ is bounded for α and β from which we see that $\widehat{f}\in\mathcal{S}$.

□

Theorem 9.2.7 (Riemann-Lebesque Lemma). *If $f\in L^1(\mathbb{R}^n)$ then $\widehat{f}$ is continuous and tends to zero as $|x|\to\infty$.*

Proof. This follows from Theorem 9.2.6 for $f\in\mathcal{S}$ and the fact that $\mathcal{S}$ is dense in L^1. Namely, if $f\in L^1$ and $\{f_j\}\subset\mathcal{S}$ such that $f_j\to f$ in L^1, then $\widehat{f_j}\to\widehat{f}$ uniformly (since $\|\widehat{f_j}-\widehat{f}\|_\infty\le\|f_j-f\|_1$) and the result follows. □

Theorem 9.2.8. *If $f(x)=e^{-a\pi|x|^2}$ with $a>0$ then $\widehat{f}(\xi)=a^{-n/2}e^{-\pi|\xi|^2/a}$*

Proof. By making the change of variables $x\to a^{-1/2}x$ we may assume that $a=1$. Since the exponential function converts sums into products, by Fubini's Theorem

$$\widehat{f}(\xi)=\int e^{-2\pi i x\cdot\xi-\pi|x|^2}\,dx=\prod_{j=1}^n\int e^{-2\pi i x_j\xi_j-\pi x_j^2}\,dx,$$

so it suffices to verify that the jth factor in the product is $e^{-\pi\xi_j^2}$. So let us consider the case $n=1$ and we have

$$\int e^{-2\pi i x\xi-\pi x^2}=e^{-\pi\xi^2}\int e^{-\pi(x+i\xi)^2}\,dx.$$

At this point we again appeal to a result that outside the scope of this class. It is known that the complex valued function of the complex variable $z=(x+iy)$ given by $\varphi(z)=e^{-\pi(z+i\xi)^2}$ is an entire function (analytic in the entire complex plane) and has the property that

$$|\varphi(z)|^2=\varphi(z)\overline{\varphi(z)}=e^{-\pi[(x+i(y+\xi))^2+(x-i(y+\xi))^2]}=e^{-2\pi x^2}\,e^{2\pi(y+\xi)^2}$$

which decreases exponentially to zero as $x \to \pm\infty$ as long as y remains bounded (recall that ξ is fixed). Thus we can apply Cauchy's Theorem to shift the contour of integration from Im $z = 0$ to Im $z = -\xi$. This gives us

$$\mathcal{F}\left(e^{-|x|^2}\right)(\xi) = e^{-\pi\xi^2}\int e^{-\pi(x+i\xi)^2}\,dx = e^{-\pi\xi^2}\int e^{-\pi x^2}\,dx = e^{-\pi\xi^2}.$$

□

Theorem 9.2.9 (Plancherel Formula for $\mathcal{S}$). *If $f, g \in \mathcal{S}$ then*

$$\int f\widehat{g} = \int \widehat{f}g$$

Proof. By Fubini's Theorem

$$\int f(x)\widehat{g}(x)\,dx = \iint f(x)g(y)e^{-2\pi i x\cdot y}\,dy\,dx = \int \widehat{f}(y)g(y)\,dy.$$

□

This result can be used to prove the following major result.

Theorem 9.2.10. *1. Let $\mathcal{F}^*f(x) = \mathcal{F}f(-x)$, then $f \in \mathcal{S}$ implies $\mathcal{F}^*\mathcal{F}f(x) = f(x)$.*

2. The Fourier transform is an isomorphism of $\mathcal{S}$ onto $\mathcal{S}$.

3. The Fourier transform extends to a unitary isomorphism of L^2 onto itself, i.e., for all $f \in L^2$ we have

$$\|f\|_2 = \|\widehat{f}\|_2.$$

This result is often refered to as ***The Plancherel Theorem****.*

Proof. 1. Given $\epsilon > 0$ and $x \in \mathbb{R}^n$, set

$$\varphi(\xi) = e^{-2\pi i x\cdot\xi - \epsilon^2\pi|\xi|^2}.$$

Then using Theorem 9.2.8 we have

$$\widehat{\varphi}(y) = \int e^{-2\pi i(y-x)\cdot\xi}\,e^{-\epsilon^2\pi|\xi|^2}\,d\xi = \epsilon^{-n}e^{-\pi|x-y|^2/\epsilon^2}.$$

Thus

$$\widehat{\varphi}(y) = \epsilon^{-n}g\left(\frac{|x-y|}{\epsilon}\right) = g_\epsilon(x-y) \quad \text{where} \quad g(x) = e^{-\ \pi|x|^2}.$$

Then by Theorem 9.2.9, we have

$$f * g_\epsilon(x) = \int f(x) g_\epsilon(x-y)\, dy = \int f\,\widehat{\varphi} = \int \widehat{f}\,\varphi = \int e^{-\pi\epsilon^2|\xi|^2}\, e^{2\pi i x\cdot\xi}\widehat{f}(\xi)\, d\xi.$$

Now by Theorems 9.2.3 and 9.2.4, $f * g_\epsilon \to f$ uniformly as $\epsilon \to 0$ since f is in $\mathcal{S}$ and functions in $\mathcal{S}$ are uniformly continuous. But for each x, by the dominated convergence theorem, we have

$$\int e^{-\pi\epsilon^2|\xi|^2}\, e^{2\pi i x\cdot\xi}\widehat{f}(\xi)\, d\xi \to \int e^{2\pi i x\cdot\xi}\widehat{f}(\xi)\, d\xi = \mathcal{F}\left(\widehat{f}\right)(x).$$

2. This is just a restatment of part 1 since for $f \in \mathcal{S}$ define $g(\xi) = \mathcal{F}(f)(-\xi) = \mathcal{F}^* f(\xi) \in \mathcal{S}$ and we have $f(x) = \mathcal{F}(g)(x)$ by part 1.

3. Since $\mathcal{S}$ is dense in L^2, by part 2 we only need to show that $\|\widehat{f}\|_2 = \|f\|_2$ for all $f \in \mathcal{S}$. If $f \in \mathcal{S}$, set $g(x) = \overline{f(-x)}$. We see that

$$\begin{aligned}
\widehat{g}(\xi) &= \int e^{-2\pi i x\cdot\xi}\, g(x)\, dx = \int e^{-2\pi i x\cdot\xi}\, \overline{f(-x)}\, dx \\
&= \overline{\int e^{2\pi i x\cdot\xi}\, f(-x)\, dx} \\
&= \overline{\int e^{-2\pi i x\cdot\xi}\, f(x)\, dx} \\
&= \overline{\widehat{f}(\xi)}.
\end{aligned}$$

Thus by Theorem 9.2.6 part 1 and Theorem 9.2.9 we have

$$\begin{aligned}
\|f\|_2^2 = \int |f(x)|^2\, dx &= \int f(x)\overline{f(x)}\, dx \\
&= (f * g)(0) = \mathcal{F}^*\mathcal{F}(f * g)(0) \\
&= \int e^{2\pi i\, 0\cdot\xi}\mathcal{F}(f * g)(\xi)\, d\xi \\
&= \int \widehat{f}(\xi)\, \overline{\widehat{f}(\xi)}\, d\xi \\
&= \|\widehat{f}\|_2^2.
\end{aligned}$$

□

9.3 The Heat Equation in $\mathbb{R}^{\mathbf{n}}$

Now we can begin our discussion of the heat equation on $\mathbb{R}^n \times \mathbb{R}_+$. We first consider the initial value problem

$$\begin{cases} \partial_t u = \Delta u \\ u(x,0) = f(x) \end{cases} \tag{9.3.1}$$

To solve this problem we first assume that $f \in \mathcal{S}$ and apply the Fourier transform to construct the so-called *Gaussian or Heat Kernel*. We have, by Theorem 9.2.6 part 3,

$$\partial_t \widehat{u}(\xi,t) + 4\pi^2|\xi|^2\widehat{u}(\xi,t) = 0, \quad \widehat{u}(\xi,0) = \widehat{f}(\xi).$$

For each ξ this is an initial value problem in t and we can solve it to get

$$\widehat{u}(\xi,t) = \widehat{f}(\xi)e^{-4\pi^2|\xi|^2 t}, \quad (t > 0).$$

Thus we obtain a product of two functions. Next we apply the inverse Fourier transform to both sides. If we apply Theorem 9.2.8 we have

$$K_t(x) \equiv \mathcal{F}^*\left(e^{-4\pi^2|\xi|^2 t}\right)(x) = \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t}. \tag{9.3.2}$$

Theorem 9.2.6 part 1 and Theorem 9.2.10 part 1, imply that the solution to our IVP is

$$u(x,t) = (K_t * f)(x) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-|x-y|^2/4t} f(y)\,dy.$$

Here we have obtained a formal solution. To prove that this is actually a solution requires either some hard estimates or the use of approximate identities and delta families to show that the initial condition is actually satisfied. Note that

$$K_t(x) = t^{-n/2}K_1(t^{-1/2}x), \quad \int_{\mathbb{R}^n} K_t(x)\,dx = \widehat{K_t(0)} = 1.$$

Therefore by using $\epsilon = t^{1/2}$ we see from Theorem 9.2.3, that $\{K_t\}_{t>0}$ is an approximation to the identity. Thus we have the main result.

Theorem 9.3.1. *If $f \in L^p(\mathbb{R}^n)$, $1 \le p \le \infty$. Then $u(x,t) = (K_t * f)(x)$ satisfies*

$$\partial_t u - \Delta u = 0, \quad \text{on } \mathbb{R}^n \times (0,\infty)$$

and

1. *If f is bounded and continuous, then u is continuous on $\mathbb{R}^n \times [0, \infty)$ and $u(x, 0) = f(x)$.*

2. *If $f \in L^p$, $p < \infty$, then $u(\cdot, t) \to f$ in L^p as $t \to 0$.*

Since K_t is rapidly decreasing along with all its derivatives, we see that we can differentiate under the integral to show that for all $t > 0$, $u \in C^\infty$. So the heat operator is instantly, infinitely smoothing. This shows that we can not reverse time in the heat equation in general, since if we had a non-smooth initial data and we went back in time to say $-t_0$ then used this function as an initial data and propagated it forward in time it would have to be smooth at time zero. We also note that $K_t * f$ actually makes sense if for $t \le T$ we would only assume that $|f(x)| \le C \exp(|x|^2/(4T))$.

As for the question of uniqueness for the heat eqaution in $\mathbb{R}^n$, we have the following result from [3].

Theorem 9.3.2. *If $u \in C(\mathbb{R}^n \times [0, \infty))$, $u \in C^2(\mathbb{R}^n \times (0, \infty))$ and u satisfies*

$$\partial_t u = \Delta u \quad \text{for} \;\; t > 0, \quad u(x, 0) = 0,$$

and if for every $\epsilon > 0$ there exists $C > 0$ such that

$$|u(x, t)| \le C e^{\epsilon |x|^2}, \qquad |\nabla_x u(x, t)| \le C e^{\epsilon |x|^2},$$

then $u(x, t) \equiv 0$.

Proof. See Folland [3] Theorem 4.4, page 144. □

To see that uniqueness does not hold for the initial value problem (9.3.1) without some conditions at infinity, consider the case $n = 1$ and let $g(t)$ be any C^∞ function on $\mathbb{R}$, then formally the series

$$u(x, t) = \sum_{k=0}^{\infty} \frac{g^{(k)}(t)\, x^{2k}}{(2k)!}$$

formally satisfies the heat equation:

$$\frac{\partial u}{\partial t} = \sum_{k=0}^{\infty} \frac{g^{(k+1)}(t)\, x^{2k}}{(2k)!} = \sum_{k=1}^{\infty} \frac{g^{(k)}(t)\, x^{2(k-1)}}{(2(k-1))!}$$

$$\frac{\partial^2 u}{\partial x^2} = \sum_{k=1}^{\infty} \frac{g^{(k)}(t)\, x^{2k-2}}{(2(k-1))!} = \sum_{k=1}^{\infty} \frac{g^{(k)}(t)\, (2k)(2k-1)x^{2k-2}}{(2k)!} = \sum_{k=1}^{\infty} \frac{g^{(k)}(t)\, x^{2(k-1)}}{(2(k-1))!},$$

and therefore

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}.$$

To produce a nonzero solution to (9.3.1) with $u(x,0) = f(x) = 0$, we give a nonzero g satisfying $g^{(k)}(0) = 0$ for $k = 0, 1, \cdots$ and so that the series converges on $\mathbb{R}^n \times [0, \infty$. Such a function is given by

$$g(t) = e^{-t^{-2}}.$$

It is worth commenting that a Theroem due to D. Widder says that if the inital data data is nonnegative, i.e., $f(x) \geq 0$, then the unique nonnegative solution is given by $u(x,t) = (K_t * f)(x)$.

Let us briefly discuss the notion of a *Fundamental Solution* for a parabolic initial value problem.

Remark 9.3.1. Suppose we want to solve a Cauchy problem

$$u_t = Lu, \quad x \in \mathbb{R}^n, \ t > 0, \tag{9.3.3}$$

$$u(x,0) = f(x), \quad x \in \mathbb{R}^n, \tag{9.3.4}$$

where L is a constant coefficient differential operator.

We consider first the "distributional" problem

$$F_t = LF, \quad x \in \mathbb{R}^n, \ t > 0, \tag{9.3.5}$$

$$F(x,0) = \delta(x), \quad x \in \mathbb{R}^n. \tag{9.3.6}$$

In (9.3.6) δ is the so-called dirac distribution which is defined by way of the requirement

$$\int_{\mathbb{R}^n} \varphi(x)\,\delta(x)\,dx = \varphi(0),$$

for every $\varphi \in C_c^\infty(\mathbb{R}^n)$. Such an F is called a *fundamental solution for the initial value problem*. We have already constructed such a function above for $L = \Delta$, namely, $F = K$.

Once we have F, the solution to (9.3.3), (9.3.4) is given by

$$u(x,t) = \int_{\mathbb{R}^n} F(x-y,t)\,f(y)\,dy \equiv (F(\cdot,t) * f)(x). \tag{9.3.7}$$

There is also a version of *Duhamel's Principle* for parabolic problems. Consider the nonhomogeneous problem

$$\begin{aligned} &u_t = Lu + g(x,t), \quad x \in \mathbb{R}^n,\ t > 0, \\ &u(x,0) = 0, \quad x \in \mathbb{R}^n. \end{aligned} \tag{9.3.8}$$

The solution to (9.3.8) is given by

$$u(x,t) = \int_0^t \int_{\mathbb{R}^n} F(x-y,t-s)\,g(y,s)\,dyds. \tag{9.3.9}$$

Just to make sure that there is no confusion with the literature, for an operator $(\partial_t - L)$ the fundamental solution of the initial value problem (9.3.5), (9.3.6) coincides with the "free space" fundamental solution which satisfies

$$(\partial_t - L)F(x,t) = \delta(x,t), \tag{9.3.10}$$

$$F(x,t) = 0, \quad t < 0. \tag{9.3.11}$$

For example, for the heat equation the free space fundamental solution is given by

$$K(x,t) = \begin{cases} K_t(x) & t > 0 \\ 0 & t < 0 \end{cases} \tag{9.3.12}$$

and we have the following result.

Theorem 9.3.3. *The kernel K in (9.3.12) is the fundamental solution for the heat operator, i.e.,*

$$(\partial_t - \Delta)F(x,t) = \delta(x)\delta(t).$$

Proof. See Folland [3] Theorem 4.6, page 146. □

9.4 The Heat Equation in Bounded Domains

For the heat equation on bounded domains we must once again consider the Boundary Value Problem (BVP). Generally we have a bounded domain $\Omega \subset \mathbb{R}^n$ and we consider a time interval $0 \leq t \leq T < \infty$. Thus the data for the problem generally consists of initial data at time $t = 0$ given as $u(x, 0)$ and some type of boundary data on $\partial\Omega \times [0, T]$. There are three physically motivated BCs often considered:

1. $u(x, t) = f(x, t)$ for $(x, t) \in \partial\Omega \times [0, T]$. Thus the temperature on the body is specified.

2. $\dfrac{\partial u}{\partial n}(x, t) = 0$ for $(x, t) \in \partial\Omega \times [0, T]$. In this case the body is insulated, i.e., there is no heat flow in or out of the region.

3. $\left(\dfrac{\partial u}{\partial n} + c(u - u_0)\right)(x, t) = 0$. This condition cooresponds to Newton's Law of Cooling in which outside the region Ω a temperature $u_0(x, t)$ is maintained and the rate of heat flow in or out of the region is proportional to $(u - u_0)(x, t)$

For parabolic BVP on bounded domains we once again have a maximum principle given in [3] by

Theorem 9.4.1. *Let Ω be a bounded domain in $\mathbb{R}^n$ and $0 < T < \infty$. Let u be a real-valued continuous function on $\overline{\Omega} \times [0, T]$ satisfying $(\partial_t - \Delta)u = 0$ on $\Omega \times (0, T)$. Then*

$$\max_{x \in \overline{\Omega} \times [0,T]} |u(x, t)| = \max \left\{ \max_{x \in \partial\Omega \times [0,T]} |u(x, t)|, \max_{x \in \Omega} |u(x, 0)| \right\}.$$

Proof. For any $\epsilon > 0$ set $v(x, t) = u(x, t) + \epsilon|x|^2$. Then $(\partial_t - \Delta)v = -2n\epsilon < 0$. Also fix any $0 < T' < T$. If the maximum of v on $\overline{\Omega} \times [0, T']$ occurs at an interior point (x_0, t_0) of $\overline{\Omega} \times [0, T']$, then from calculus we must have $\partial_{x_j} v(x_0, t_0) = 0$ and $\partial_t v(x_0, t_0) = 0$ and the second derivatives $\partial^2_{x_j} v(x_0, t_0) \leq 0$. Thus we must have $(\partial_t - \Delta)v(x_0, t_0) \geq 0$ which contradicts the result $(\partial_t - \Delta)v(x_0, t_0) = -2n\epsilon < 0$.

If the maximum occurs on $\Omega \times \{T'\}$, then at the maximum $\partial_t v \geq 0$ and $\Delta v \leq 0$ so that, just as above we have a contradiction. Thus we have

$$\begin{aligned}
\max_{x \in \overline{\Omega} \times [0,T']} |u(x, t)| &\leq \max_{x \in \overline{\Omega} \times [0,T']} |v(x, t)| \\
&= \max_{x \in (\Omega \times \{0\}) \cup (\partial\Omega \times [0,T'])} |v(x, t)| \\
&= \max_{x \in (\Omega \times \{0\}) \cup (\partial\Omega \times [0,T'])} |u(x, t)| + \epsilon \max_{x \in \overline{\Omega}} |x|^2.
\end{aligned}$$

Passing to the limits as $\epsilon \to 0$ and $T' \to T$ gives the desired result. We note that T' was needed because we can only apply the calculus results in the case that u is smooth which we don't know at $t = T$.

$\square$

Remark 9.4.1. By replacing u by $-u$ we can also see that the minimum is also achieved either on $(\partial\Omega \times [0, T])$ or $(\Omega \times \{0\})$.

Corollary 9.4.1. *There is at most one continuous function u on $\overline{\Omega} \times [0, T]$ solving the BVP*

$$\begin{aligned} &u_t - \Delta u = 0 \quad (x, t) \in \Omega \times (0, T), \\ &u(x, 0) = f(x) \quad x \in \Omega, \\ &u(x, t) = g(x, t) \quad (x, t) \in \partial\Omega \times [0, T]. \end{aligned}$$

As you might expect the method of separation of variables once again plays a central role. We seek simple solutions of the heat equation in the form

$$u(x, t) = X(x)T(t)$$

which gives

$$\frac{\dot{T}(t)}{T(t)} = \frac{\Delta X(x)}{X(x)} = -\lambda^2.$$

We conclude that

$$\dot{T}(t) + \lambda^2 T(t) = 0$$

and

$$\Delta X(x) + \lambda^2 X(x) = 0, \quad X(x)\big|_{x \in \partial\Omega} = 0.$$

The first of these problems, an ordinary differential equation has general solution

$$T(t) = A \exp(-\lambda^2 t).$$

At this point recall a general result first expressed in Chapter 7.

Theorem 9.4.2. *The Laplacian operator Δ on a bounded domain Ω with smooth boundary $\partial\Omega$ admits an orthonormal basis for $L^2(\Omega)$ consisting of eigenfunctions $\psi_j(x)$ with associated eigenvalues $(-\lambda_j^2) < 0$ satisfying*

$$\begin{aligned} &\Delta\psi_j + \lambda_j^2 \psi_j = 0, \\ &\psi(x) = 0 \quad x \in \partial\Omega. \end{aligned}$$

Thus $\langle \psi_j, \psi_k \rangle = \delta_{j,k}$ where $< \cdot, \cdot >$ is the inner product in $L^2(\Omega)$ given by

$$\langle f, g \rangle = \int_\Omega f(x)\overline{g(x)}\, dx$$

where $\overline{z}$ denotes the complex conjugate of z.

In particular, for every $f \in L^2(\Omega)$ we have

$$f = \sum_{j=1}^{\infty} \langle f, \psi_j \rangle \psi_j,$$

in the sense of $L^2(\Omega)$.

Similar results hold for the Neumann boundary conditions ($\partial_n u = 0$) and the radiation conditions ($\partial_n u + cu) = 0$.

With this result we can write the solution to

$$\begin{aligned} &u_t - \Delta u = 0 \quad (x,t) \in \Omega \times (0,T), \\ &u(x,0) = f(x) \quad x \in \Omega, \\ &u(x,t) = 0 \quad (x,t) \in \partial\Omega \times [0,T]. \end{aligned}$$

as

$$u(x,t) = \sum_{j=1}^{\infty} e^{-\lambda_j^2 t} f_j \psi_j(x),$$

where

$$f(x) = \sum_{j=1}^{\infty} f_j \psi_j(x),$$

and

$$f_j = \langle f, \psi_j \rangle.$$

Exercise Set 5: Parabolic Equations

1. Consider the nonhomogeneous heat problems:

   (a) $u_t = u_{xx} + f(x,t), \quad x \in \mathbb{R},\ t > 0, \quad u(x,0) = 0, \quad x \in \mathbb{R}$. Apply a Duhamel's principle to find the following formula for the solution of this problem:

   $$u(x,t) = \int_0^t \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi(t-\tau)}} e^{-(x-\xi)^2/(4(t-\tau))} f(\xi,\tau)\, d\xi d\tau.$$

   (b) Solve $u_t = 4u_{xx} + t + e^t, \quad x \in \mathbb{R},\ t > 0, \quad u(x,0) = 2, \quad x \in \mathbb{R}$.

   (c) Solve $u_t = u_{xx} + e^{-t}\cos(x), \quad x \in \mathbb{R},\ t > 0, \quad u(x,0) = \cos(x), \quad x \in \mathbb{R}$.

   (d) Solve $u_t = u_{xx} + e^t \sin(x), \quad x \in \mathbb{R},\ t > 0, \quad u(x,0) = \sin(x), \quad x \in \mathbb{R}$.

2. The heat equation in the semi-infinite rod with its end kept at zero temperature or being insulated leads to the initial-boundary value problems

$$u_t = u_{xx}, \quad 0 < x < \infty,\ t > 0$$

$$u(x, 0) = \phi(x), \quad 0 \le x < \infty$$

$$u(0, t) = 0, \quad t \ge 0 \tag{1}$$

or

$$u_x(0, t) = 0 \quad t \ge 0. \tag{2}$$

For the boundary condition (1) show that

$$u(x, t) = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{1}{\sqrt{t}} \left[\exp\left(\frac{-(x-\xi)^2}{4t} \right) - \exp\left(\frac{-(x+\xi)^2}{4t} \right) \right] \phi(\xi)\ d\xi.$$

For the boundary condition (2) show that

$$u(x, t) = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{1}{\sqrt{t}} \left[\exp\left(\frac{-(x-\xi)^2}{4t} \right) + \exp\left(\frac{-(x+\xi)^2}{4t} \right) \right] \phi(\xi)\ d\xi.$$

3. Recall that the solution to the initial value heat problem

$$u_t = u_{xx}, \quad -\infty < x < \infty,\ t > 0$$

$$u(x, 0) = f(x)$$

is given by

$$u(x, t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} \exp\left(\frac{-(x-y)^2}{4t} \right) f(y) dy.$$

(a) Prove that the solution depends continuously on the data in the sense that if

$$|f(x) - \tilde{f}(x)| < \epsilon, \qquad x \in \mathbb{R},$$

then the corresponding solutions satisfy

$$|u(x, t) - \tilde{u}(x, t)| < \epsilon, \quad x \in \mathbb{R},\ t > 0.$$

(b) Assume that $f(x)$ is continuous and bounded. Show that

$$\lim_{t\to 0^+} u(x,t) = f(x)$$

by first proving the following result.

Lemma: Let ψ be positive, continuous and integrable on $(-\infty,\infty)$ and satisfy

$$\int_{-\infty}^{\infty} \psi(r)dr = 1.$$

If f is continuous and bounded, then

$$\lim_{\delta\to 0} \frac{1}{\delta}\int_{-\infty}^{\infty} \psi\left(\frac{y-x}{\delta}\right) f(y)dy = f(x).$$

Hint: Make the obvious change of variable and reduce the problem to showing

$$\lim_{\delta\to 0}\int_{-\infty}^{\infty} \psi(r)\left[f(x+\delta r) - f(x)\right]dr = 0.$$

Prove (b) by applying the Lemma to

$$\psi(x) = \frac{1}{\sqrt{\pi}} e^{-x^2}.$$

4. Consider the initial boundary value problem,

$$\begin{array}{ll} u_t = u_{xx} + F(x,t), & 0 < x < l,\ t > 0 \\ u(x,0) = 0, & 0 \le x \le l \\ u(0,t) = 0, \quad u(l,t) = 0, & t \ge 0. \end{array}$$

Use Duhamel's principle and a formal series solution to obtain the following formula for the solution of the nonhomogeneous problem,

$$u(x,t) = \sum_{k=1}^{\infty}\left\{\int_0^t F_k(s) e^{-\frac{k^2\pi^2(t-s)}{l^2}}\, ds\right\}\sin\frac{k\pi x}{l}$$

where

$$F_k(s) = \frac{2}{l}\int_0^l F(x,s)\sin\frac{k\pi x}{l}\, dx.$$

If $F(x,t) = F(t)\sin(\pi x/l)$ show that the solution is

$$u(x,t) = \left(\int_0^t F(s) e^{\frac{\pi^2 s}{l^2}}\, ds\right)\sin\frac{\pi x}{l} e^{-\frac{\pi^2 t}{l^2}}$$

5. (a) Separate variables to construct a series solution of

$$u_t = u_{xx},\ 0 < x < \pi,\ t > 0$$

$$u(x,0) = x,\ 0 \le x < \pi$$

$$u_x(0,t) = 0 = u_x(\pi,t),\ t \ge 0$$

   (b) Carefully justify that the series solution satisfies the boundary conditions and the initial condition and show that for each $t > 0$ the function $u(x,t)$ defined by this series represents a C^∞ function in x that satisfies the heat equation.

6. Show that the solution to the initial value problem

$$\begin{aligned} u_{xx} &= u_t, && x \in \mathbb{R},\ t > 0 \\ u(x,0) &= e^{bx}, && x \in \mathbb{R} \end{aligned}$$

is given by $u(x,t) = e^{bx}e^{b^2 t}$

7. Solve the heat problem

$$\begin{aligned} &u_t = u_{xx}, \quad 0 < x < 1,\ t > 0 \\ &u_x(0,t) = 1, \quad u_x(1,t) = 0, \\ &u(x,0) = 0. \end{aligned}$$

8. Do both parts

   (a) Let $\{u_j(y,t)\}_{j=1}^n$ be solutions of the one-dimensional heat equation $u_t = u_{yy}$. Show that

$$v(x,t) = \prod_{j=1}^n u_j(x_j,t)$$

   satisfies the n-dimensional heat equation. What is special about the heat equation that makes this true?

   (b) Suppose that $\{u_k(x,t)\}_{k=1}^n$ are solutions of the nonhomogeneous heat problems

$$u_t = u_{xx}, \quad u(x,t) = f_k(x_k), \quad x \in \mathbb{R}^n,\ t > 0,\ k = 1, \cdots, n.$$

Show that

$$u(x,t) = \prod_{k=1}^n u_k(x,t)$$

is a solution of the problem

$$u_t = \Delta u, \quad u(x,0) = \prod_{k=1}^n f_f(x_k).$$

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