Chapter 5

Sturm-Liouville Theory

5.1 Oscillation and Separation Theory

Consider the differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \tag{5.1.1}$$

where $a_2(x)$ is not zero for all $x \in [a, b]$, $a_i(x) \in C[a, b]$. Rewrite (5.1.1) in the form

$$y'' + \frac{a_1}{a_2}y' + \frac{a_0}{a_2}y = y'' + p(x)y' + q(x)y = 0$$

Define

$$k(x) = e^{\int p(s)ds},$$

Then

$$\frac{d}{dx}(k(x)y') + k(x)q(x)y = 0$$

or

$$(ky')' + g(x)y = 0 \tag{5.1.2}$$

Define the differential operators

$$L(y) = (ky')' + g(x)y$$

$$M(y) = a_2y'' + a_1y' + a_0y \tag{5.1.3}$$

The adjoint of M is defined by

$$\overline{M}(y) = (a_2y)'' - (a_1y)' + a_0y$$

$$\overline{M}(y) = a_2 y'' + (2a_2' - a_1)y' + (a_2'' - a_1' + a_0)y.$$

After some manipulation it is easy to show that

$$vM(u) - u\overline{M}(v) = [(a_1 - a_2')vu + a_2(vu' - uv')]'.$$

This result is called *LaGrange's identity* and we rewrite it as

$$vM(u) - u\overline{M}(v) = \frac{d}{dx}P(u,v).$$

By an integration we obtain *Green's formula* ,

$$\int_a^b \left[vM(u) - u\overline{M}(v)\right]\, dx = P(u,v)|_a^b$$

If $\overline{M}(u) = M(u)$, the equation $M(u) = 0$ is said to be self-adjoint. Hence $M(u) = 0$ is self-adjoint if

$$a_2' = a_1.$$

In this case Lagrange's identity becomes

$$vM(u) - u\overline{M}(v) = [a_2(vu' - uv')]' = [a_2(x)N(v,u)]'$$

and

$$M(u) = (a_2 u')' + a_0 u$$

Clearly the operator L defined by (5.1.3) is self-adjoint and the discussion preceding (5.1.2) shows every general linear equation can be put into self adjoint form.

5.1.1 Separation theorems

Theorem 5.1.1. *[Sturm Separation Theorem]*

1. *A nontrivial solution of $M(y) = 0$ can have at most a finite number of zeros on $[a, b]$.*
2. *All zeros of a solution are simple.*
3. *If $u_1(x)$, $u_2(x)$ are linearly independent solutions of $M(y) = 0$ then between any two zeros of $u_1(x)$ there is precisely one zero of $u_2(x)$.*

Proof. 1. Suppose there exists infinitely many zeros,$\{z_n\}$, select a subsequence $\{x_n\}$ such that $x_n \to \widehat{x}$. Then

$$0 = \lim_{n\to\infty} y(x_n) = y(\widehat{x}).$$

Also,

$$y'(\widehat{x}) = \lim_{n\to\infty} \frac{y(x_n) - y(\widehat{x})}{x_n - \widehat{x}} = 0$$

and so $y(x) = 0$ by uniqueness.

2. The proof of this was a much earlier exercise.

3. Suppose x_0, x_1 are consecutive zeros of $u_1(x)$, and assume that$x_0 < x_1$. Then $u_2(x_0) \neq 0$ and $u_2(x_1) \neq 0$, or else $W(u_1, u_2)(x_i) = 0, i = 0, 1$. So without loss of generality assume $u_1(x) > 0, x \in (x_0, x_1), u_2(x_0) > 0$. Now

$$W(u_1, u_2)(x_0) = -u_1'(x_0)u_2(x_0)$$

$$W(u_1, u_2)(x_1) = -u_1'(x_1)u_2(x_1)$$

Since $u_1'(x_0) > 0, W(x_0) < 0$. Because the Wronskian of u_1, u_2 cannot change sign, $W(x_1) < 0$. But $u_1'(x_1) < 0$ so this would require that $u_2(x_1) < 0$. Hence $u_2(x)$ must vanish in (x_0, x_1).

If we apply the argument with the roles of u_1 and u_2 interchanged, we see that between two consecutive zeros of u_2 there must be a zero of $u_1(x)$. Hence the zeros of u_1 and u_2 must interface.

□

Roughly speaking, the Sturm Separation theorem states that linearly independent solutions have the same number of zeros. If we consider two different equations, for example

$$y'' + y = 0, \quad y'' + 4y = 0$$

then solutions of the second equation oscillate more rapidly than those of the first. More generally, Sturm Comparison theorems address the rate of oscillation of solutions of different equations.

5.1.2 Oscillation Theory

Here we shall consider equations of the form

$$L(y) = (ky')' + gy = 0, a < x < b$$

where $k \in C^1(a, b), g \in C^0[a, b]$, and $k > 0$ on $[a, b]$.

Theorem 5.1.2. *Let $L_i(y) = (ky')' + g_i y = 0$ where $g_2 > g_1$ and g_2 is not identically equal to g_1 on any subinterval of (a, b). If $L_1(u_1) = 0$ and $L_2(u_2) = 0$, then between any two consecutive zeros of $u_1(x)$ there is a zero of $u_2(x)$.*

Proof. Suppose $u_1(x_1) = u_1(x_2) = 0$ and $u_2(x) \neq 0$ on (x_1, x_2). Without loss of generality take $u_1, u_2 > 0$ on (x_1, x_2) Lagrange's identity gives

$$u_2 L_1(u_1) - u_1 L_1(u_2) = \frac{d}{dx}(k(x)(u_2 u_1' - u_1 u_2').$$

We also have

$$L_1(u_2) - L_2(u_2) = (g_1 - g_2)u_2.$$

Hence

$$u_2 L_1(u_1) - u_1[L_2(u_2) + (g_1 - g_2)u_2] = (k(x)(u_2 u_1' - u_1 u_2'))'$$

or

$$k(x)(u_2 u_1' - u_1 u_2')|_{x_1}^{x_2} = \int_{x_1}^{x_2} (g_2 - g_1)u_1 u_2 dx > 0.$$

However, the left hand side reduces to

$$k(x_2)u_2(x_2)u_1'(x_2) - k(x_1)u_2(x_1)u_1'(x_1). \qquad (*)$$

Since $u_2(x_2) \geq 0, u_1'(x_2) < 0$ and $u_2(x_1) \geq 0, u_1'(x_1) > 0$ the above expression is nonpositive and hence we obtain a contradiction. □

Suppose that we assumed $u_2(x_1) = 0$, then in the above proof expression $(*)$ becomes

$$k(x_2)u_1'(x_2)u_2(x_2).$$

If $u_2(x_2) \geq 0$, then again $(*) \leq 0$ and a contradiction would be obtained. Thus Theorem 5.1.2 could be restated: If $k \in C^1(a,b), g \in C^0[a,b], k > 0$ on $[a,b]$ and $u_1(a) = u_2(a) = 0$ and $u_1(x_1) = 0, a < x_1 < b$, then there exists $z, a < z < x_1$ such that $u_2(z) = 0$. Thus $u_2(x)$ has at least as many zeros as $u_1(x)$ on $[a,b]$.

A more general version of this theorem is

Theorem 5.1.3. *Assume $p, q \in C^0[a,b]$ and $z(x)$ is a non trivial solution of*

$$z'' + q(x)z = 0$$

where

$$z(a) = z(b) = 0.$$

If

$$\int_a^b (p - q)z^2 dx \geq 0$$

then a nontrivial solution of

$$y'' + p(x)y = 0$$
$$y(a) = 0$$

has a zero in the interval $(a, b]$.

Proof. Suppose $y(x) \neq 0$ in $(a, b]$. Then

$$z(z'' + qz) = 0$$

$$\frac{z^2}{y}(y'' + py) = 0$$

or

$$zz'' - z^2\frac{y''}{y} = z^2(p - q)$$

or

$$\frac{z}{y}(yz' - zy')' = z^2(p - q).$$

Now note that

$$\lim_{x \to a} \frac{z(x)}{y(x)} = \lim_{z \to a} \frac{z'(a)}{y'(a)}$$

and since $z'(a) \neq 0, y'(a) \neq 0, z', y' \in C[a, b]$, this limit exists and is finite. Hence it makes sense to write

$$\int_a^b \frac{z}{y}(yz' - zy')'dx = \int_a^b z^2(p - q)dx \geq 0.$$

Now integrate by parts and since $z(b) = 0, y(b) \neq 0, z(a) = y(a) = 0$,

$$\frac{z}{y}(yz' - zy')|_a^b - \int_a^b (yz' - zy')(\frac{z}{y})' = \int_a^b z^2(p - q)dx \geq 0, \quad (*)$$

or

$$-\int_a^b \frac{(yz' - zy')^2}{y^2}dx \geq 0$$

or

$$0 \geq \int_a^b \frac{(yz' - zy')^2}{y^2}dx.$$

The right hand side is identically zero if $y(x) = cz(x)$ in which case the result is trivially true. So if $y(x) \neq cz(x)$, the right hand side is positive and we get a contradiction. Hence $y(x)$ must vanish in $(a, b]$. □

The proof shows that if $p(x) \neq q(x)$ then

$$\int_a^b z^2(p-q)dx > 0.$$

In this case $y(x)$ must have a zero in (a,b). If not, then just as before we could derive (*) by dividing by $y(x)$ and the boundary term in (*) would vanish since $y(b) = 0$, and we would obtain

$$\int_a^b \frac{(yz' - zy')^2}{y^2} dx < 0,$$

which is a contradiction.

We conclude with a generalization of these results.

Theorem 5.1.4. *Let $k_i \in C^1[a,b], g_i \in C[a,b]$ with $k_i > 0$. If z is a nontrivial solution of*

$$(k_1 z)' + g_1 z = 0$$
$$z(a) = z(b) = 0,$$

and y is a non trivial solution of

$$(k_2 y')' + g_2 y = 0$$
$$y(a) = 0$$

and

$$\int_a^b (k_1 - k_2)(z')^2 + (g_2 - g_1)z^2 dx \geq 0,$$

then $y(x)$ has a zero in $(a,b]$. If the inequality is strict, the zero is in the open interval (a,b).

Proof. Multiply the first equation by z and subtract the second multiplied by (z^2/y) to obtain the Picone formula,

$$\int_a^b (k_1 - k_2)(z')^2 + (g_2 - g_1)z^2 dx + \int_a^b k_2 (\frac{yz' - zy'}{y})^2 dx = \frac{z}{y}(k_1 y z' - k_2 y' z)|_a^b$$

and proceed as before. □

As an immediate consequence of this theorem we obtain

Theorem 5.1.5. *[Sturm-Picone Theorem] Suppose*

$$(k_1 z')' + g_1 z = 0$$
$$(k_2 y')' + g_2 y = 0$$

where $g_2 \geq g_1$ and $k_1 \geq k_2 > 0$ and $g_2 \not\equiv g_1, k_2 \not\equiv k_1$ on $[a,b]$ and

$$z(a) = z(b) = 0.$$

then $y(x)$ has a zero in (a,b).

5.2 Boundary Value Problems

We consider the problem of solving

$$M(y) = a_2 y'' + a_1 y' + a_0 y = f(x), \quad a < x < b \tag{5.1.4}$$

subject to the boundary conditions

$$B_1(y) = \alpha_{11} y(a) + \alpha_{12} y'(a) + \beta_{11} y(b) + \beta_{12} y'(b) = \gamma_1 \tag{5.1.5}$$

$$B_2(y) = \alpha_{21} y(a) + \alpha_{22} y'(a) + \beta_{21} y(b) + \beta_{22} y'(b) = \gamma_2.$$

Here $a_2, a_1, a_0 \in C[a,b], a_2(x) \neq 0, \alpha_{ij}, \beta_{ij}, \gamma_i$ are constants. Equations (5.1.4) and (5.1.5) constitute what is called a boundary value problem(BVP). If $\beta_{11} = \beta_{12} = 0$ and $\alpha_{21} = \alpha_{22} = 0$, the boundary conditions are separated. If $\alpha_{12} = \beta_{12} = 0$ and $\alpha_{21} = \beta_{21} = 0$, and $\gamma_1 = \gamma_2 = 0$ the boundary conditions are periodic. If $\alpha_{12} = \beta_{11} = \beta_{12} = \alpha_{21} = \beta_{21} = \beta_{22} = 0$ we have the initial conditions $y(a) = \frac{\gamma_1}{\alpha_{11}}, y'(a) = \frac{\gamma_2}{\alpha_{21}}$. Note that the boundary operators B_i are linear. We refer to $\{f(x), \gamma_1, \gamma_2\}$ as the data of the BVP.

Theorem 5.1.1. *The BVP (5.1.4)-(5.1.5) with data $\{0, \gamma_1, \gamma_2\}$ has a unique solution iff the BVP with data $\{0, 0, 0\}$ has only the trivial solution.*

Proof. Assume first that we know that solutions to the BVP (5.1.4), (5.1.5) are unique and suppose y_1, y_2 are linearly independent solutions of (5.1.4). If u is any solution of $M(y) = 0$, with data $\{0, \gamma_1, \gamma_2\}$, then, since every solution must be a linear combination of y_1 and y_2, there exist unique α_1, α_2 so that

$$u = \alpha_1 y_1 + \alpha_2 y_2.$$

Then

$$\alpha_1 B_1(y_1) + \alpha_2 B_1(y_2) = \gamma_1$$

$$\alpha_1 B_2(y_1) + \alpha_2 B_2(y_2) = \gamma_2,$$

or

$$\begin{pmatrix} B_1(y_1) & B_1(y_2) \\ B_2(y_1) & B_2(y_2) \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \tag{5.1.6}$$

Since α_1, α_2 are unique,

$$\det \begin{pmatrix} B_1(y_1) & B_1(y_2) \\ B_2(y_1) & B_2(y_2) \end{pmatrix} \neq 0. \tag{5.1.7}$$

Now let us suppose that w solves the boundary value problem with data $\{0,0,0\}$ and we write

$$w = \xi_1 y_1 + \xi_2 y_2 \tag{5.1.8}$$

then as above we get

$$\begin{pmatrix} B_1(y_1) & B_1(y_2) \\ B_2(y_1) & B_2(y_2) \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and since the coefficient matrix is nonsingular we must have $\xi_1 = \xi_2 = 0$ by (5.1.7).

Conversely, if the BVP with data$\{0,0,0\}$ has only the trivial solution then from (5.1.8) we conclude that (5.1.7) holds and hence (5.1.6) has a unique solution. □

Theorem 5.1.2. *The BVP*

$$M(y) = f, \quad x \in (a,b)$$

$$B_1(y) = \gamma_1, B_2(y) = \gamma_2$$

has a unique solution if the BVP with data $\{0,0,0\}$ *has only the trivial solution.*

Proof. Let u_1, u_2 be linearly independent solutions of $M(y) = 0$ and let y_p be a particular solution of $M(y) = f$. Seek a solution of the BVP in the form

$$y = \alpha_1 u_1 + \alpha_2 u_2 + y_p.$$

Solve for α_1, α_2 such that

$$B_1(y) = \alpha_1 B_1(u_1) + \alpha_2 B_1(u_2) + B_1(y_p) = \gamma_1$$

$$B_2(y) = \alpha_1 B_2(u_1) + \alpha_2 B_2(u_2) + B_2(y_p) = \gamma_2$$

or

$$\begin{pmatrix} B_1(u_1) & B_1(u_2) \\ B_2(u_1) & B_2(u_2) \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} \gamma_1 - B_1(y_p) \\ \gamma_2 - B_2(y_p) \end{pmatrix}.$$

Since the homogeneous problem has only the trivial solution the matrix is invertible and so

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} B_1(u_1) & B_1(u_2) \\ B_2(u_1) & B_2(u_2) \end{pmatrix}^{-1} \begin{pmatrix} \gamma_1 - B_1(y_p) \\ \gamma_2 - B_2(y_p) \end{pmatrix}.$$

□

5.2 Sturm-Liouville Boundary Value Problems

In paractice one often encounters a second order differential equation in so-called self-adjoint form and generally one finds that the most common boundary conditions are either separated or periodic. A second order operator L is in self-adjoint form if

$$L(y) = (ky')' + g(x)y.$$

We are particularly interested in BVP's of the form

$$L(y) + \lambda p(x)y = 0, \quad a < x < b, \tag{5.2.9}$$
$$B_1(y) = 0, \quad B_2(y) = 0, \tag{5.2.10}$$

where k, k', g, p are real and continuous on $[a, b]$, and $k, p \; > 0$ on $[a, b]$. The correponding separated boundary conditions are given by

$$B_1(y) = \alpha_1 y(a) + \alpha_2 y'(a) = 0 \tag{5.2.11}$$
$$B_2(y) = \beta_1 y(b) + \beta_2 y'(b) = 0. \tag{5.2.12}$$

The BVP (5.2.9)-(5.2.12) is called a Regular Sturm-Liouville Eigenvalue Problem. The values of λ for which the BVP has a nontrivial solution are called eigenvalues.

For a general $Ly = a_0 y'' + a_1 y' + a_2 y$, the BVP

$$L(y) + \lambda p(x)y = 0$$

$$B_1(y) = 0, B_2(y) = 0$$

is said to be self-adjoint provided

$$\int_a^b [uL(v) - vL(u)]dx = 0$$

for all u, v that satisfy the above boundary conditions (5.2.11)-(5.2.12).

Theorem 5.2.1. *The BVP corresponding to the regular SLBVP*

$$L(u) + \lambda pu = 0$$

$$\alpha_1 y(a) + \alpha_2 y'(a) = 0$$

$$\beta_1 y(b) + \beta_2 y'(b) = 0$$

is self-adjoint.

Proof. Integration by parts yields the so-called Green's formula

$$\int_a^b [vL(u) - uL(v)]dx = k(x)\,(u'v - v'u)\,|_a^b \equiv P(u,v)|_a^b \tag{5.2.13}$$

If u and v satisfy the boundary conditions at $x = a$, then

$$\begin{pmatrix} v(a) & v'(a) \\ u(a) & u'(a) \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since $\alpha_1^2 + \alpha_2^2 \neq 0$, we have

$$u'(a)v(a) - v'(a)u(a) = 0.$$

In the same way we see that if u and v satisfy the conditions at $x = b$, then

$$u'(b)v(b) - v'(b)u(b) = 0.$$

From this we see that $P(u,v)|_a^b = 0$ and the result follows. □

Note that the above proof shows, in general, that if u, v satisfy separated B.C's, then $P(u,v)|_a^b = 0$. The next theorem states that "eigenfunctions corresponding to different eigenvalues are orthogonal with respect to the weight $p(x)$."

Theorem 5.2.2. *Let $(\lambda, u), (\mu, v)$ denote an eigenpair of the RSLBVP,*

$$L(y) + \lambda p(x)y = 0$$

$$\alpha_1 y(a) + \alpha_2 y'(a) = 0$$
$$\beta_1 y(b) + \beta_2 y'(b) = 0.$$

Then

$$(\lambda - \mu)\int_a^b u(x)v(x)p(x)dx = 0,$$

i.e., u and v are orthogonal with respect to the weight p.

Proof. From Green's formula (5.2.13), we know that

$$(\lambda - \mu)\int_a^b uvp(x)dx = \int_a^b [u(L(v) - v(L(u)]\,dx =$$
$$= k(x)\,(u'v - v'u)\,|_a^b = 0.$$

□

Theorem 5.2.3. *The eigenvalues of the RSLBVP are real.*

Proof. If

$$L(u) + \lambda pu = 0,$$

then

$$\overline{L(u)} + \overline{\lambda pu} = 0,$$

or

$$L(\overline{u}) + \overline{\lambda} p\overline{u} = 0$$

and clearly

$$B_1(\overline{y}) = 0 = B_2(\overline{y}).$$

Thus $(\overline{\lambda}, \overline{\mu})$ is an eigenpair and so

$$(\lambda - \overline{\lambda}) \int_a^b u\overline{u}pdx = 0,$$

or

$$(\lambda - \overline{\lambda}) \int_a^b |u|^2 pdx = 0.$$

Since $p > 0$, and $|u|^2 \not\equiv 0$ (since u is an eigenfunction), we must have $\lambda = \overline{\lambda}$. □

An eigenvalue λ is said to be simple if the dimension of the null space of L_λ is one, i.e., the diemnsion of $\{\varphi : L_\lambda \varphi = 0\}$ is one. Otherwise λ is a multiple eigenvalue.

Theorem 5.2.4. *The eigenvalues of the RSLBVP are simple.*

Proof. Suppose u, v are eigenfunctions corresponding to the same eigenvalue λ. Then

$$\begin{pmatrix} u(a) & v(a) \\ u'(a) & v'(a) \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

where u, v satisfy

$$L(y) + \lambda py = 0.$$

Since $\alpha_1^2 + \alpha_2^2 \neq 0$, the determinant of the coefficient matrix must be zero (i.e., the homogeneous equation has nonzero solutions) and hence u, v must be linearly dependent, i.e, $v(x) = cu(x)$. □

If $k(a) = k(b)$ and instead of the separated boundary conditions, we consider the periodic boundary conditions,

$$y(a) = y(b), \quad y'(a) = y'(b),$$

then Theorems 5.2.1-5.2.3 are still true Theorem 5.2.4 is no longer true.

We have yet to address whether the RSLBVP has any eigenvalues. The following examples indicate that there are infinite, but a countable number, of eigenvalues.

Example 5.2.5. *Find the eigenvalues and eigenfunctions for*

$$y'' + \lambda y = 0, \quad 0 < x < l$$

with the boundary conditions

1. $y(0) = 0, \quad y(l) = 0.$

   If $\lambda = 0$, $y(x) = ax + b$ *and the boundary conditions imply* $a = b = 0$. *If* $\lambda < 0$, *say* $\lambda = -\mu^2$, *then*

   $$y(x) = a\sinh(\mu x) + b\cosh(\mu x).$$

   Then $y(0) = 0$ *only if* $b = 0$ *and* $y(l) = 0$ *only if* $a = 0$.

   If $\lambda > 0$, *say* $\lambda = \mu^2$, *then*

   $$y(x) = a\sin(\mu x) + b\cos(\mu x).$$

   and $y(0) = 0$ *if* $b = 0$. *To satisfy the boundary condition at* $x = l$ *we need*

   $$a\sin(\mu l) = 0.$$

   We get a nontrivial solution if

   $$\mu l = n\pi, \quad n = \pm 1, \pm 2, \pm 3.....$$

   Hence the eigenvalues and eigenfunctions are given by

   $$\lambda_n = (n\pi/l)^2, \quad y_n(x) = \sin(n\pi x/l), \quad n = 1, 2, 3, ...$$

2. $y(0) = 0, \quad y'(l) = 0$

   As above it is easy to verify that eigenvalues must be positive. If $\lambda = \mu^2 > 0$, *then*

   $$y(x) = a\sin(\mu x) + b\cos(\mu x).$$

   and $y(0) = 0$ *if* $b = 0$. *The second boundary condition gives*

   $$a\mu\cos(\mu l) = 0$$

   or

   $$\mu l = \left(n + \frac{1}{2}\right)\pi, \quad n = 0, \pm 1, \pm 2, \cdots$$

   or

   $$\lambda_n = \left(\left(n + \frac{1}{2}\right)\frac{\pi}{l}\right)^2, \quad n = 0, \pm 1, \pm 2, \cdots$$

   and

   $$y_n(x) = \sin\left(\left(n + \frac{1}{2}\right)\frac{\pi}{l}x\right).$$

3. $y'(0) = y'(l) = 0$
If $\lambda = 0$, *then* $y = ax + b$ *and* $y'(0) = y'(l) = 0$ *if* $a = 0$. *Hence the constant function is an eigenfunction corresponding to the eigenvalue 0. It is easy to verify than an eigenvalue for this problem cannot be negative. If* $\lambda = \mu^2 > 0$, *then* $y(x) = a\cos\mu x + b\sin\mu x$ *and* $y'(0) = 0$ *if* $b = 0$. *Then*

$$y'(l) = -a\mu\sin\mu l = 0$$

if

$$\mu l = n\pi, \quad n = 0, \pm 1, \pm 2, \cdots$$

Hence eigenvalues and eigenfunctions are given by

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \; y_n(x) = \sin\left(\frac{n\pi}{l}x\right), n = 0, 1, 2, \cdots$$

4. $y(0) + y'(0) = 0, \quad y(l) = 0$
If $\lambda = -\mu^2 < 0$, $\mu > 0$ *then*

$$y(x) = ae^{\mu x} + be^{-\mu x}$$

and the boundary conditions require that

$$y(0) + y'(0) = (a + b) + \mu(a - b) = 0$$
$$y(l) = ae^{\mu l} + be^{-\mu l} = 0$$

which implies $b = \exp(2\mu l)a$ *and*

$$a[(1 - e^{2\mu l}) + \mu(1 + e^{2\mu l})] = 0.$$

This can be written as

$$e^{2\mu l} = \frac{1 - \mu}{1 + \mu}$$

and by graphing each side it is easy to see that this equation is satisfied only when $\mu = 0$. *Thus we have* $a = 0$ *and hence* $b = 0$. *If* $\lambda = 0$, $y = ax + b$ *and the boundary conditions require that*

$$y(0) + y'(0) = b + a = 0$$
$$a(l - 1) = 0$$

If $l = 1$, *then* a *is arbitrary and an eigenfunction is* $y(x) = x - 1$. *If* $l \neq 1$, *then* $a = 0$ *and hence* $b = 0$ *and so* $\lambda = 0$ *is not an eigenvalue. If* $\lambda = \mu^2 > 0$ *then* $y = a\cos\mu x + b\sin\mu x$ *and to satisfy the boundary conditions we need*

$$a + \mu b = 0, \quad a\cos\mu l + b\sin\mu l = 0$$

or

$$\tan \mu l = \mu.$$

It is easy to see that there are infinitely many eigenvalues λ_n that satisfy

$$\sqrt{\lambda_n} = \tan \sqrt{\lambda_n} l$$

with corresponding eigenfunctions

$$y_n(x) = \sin \sqrt{\lambda_n} x - \sqrt{\lambda_n} \cos \sqrt{\lambda_n} x.$$

5. $y(0) = y(l), \quad y'(0) = y'(l)$

If $\lambda = -\mu^2 < 0$, then

$$y(x) = a \sinh \mu x + b \cosh \mu x$$

and

$$y(0) = b = y(l) = a \sinh \mu l + b \cosh \mu l$$

while

$$y'(0) = a\mu = y'(l) = a\mu \cosh \mu l + b\mu \sinh \mu l$$

or

$$\begin{pmatrix} \sinh \mu l & \cosh \mu l - 1 \\ \mu(\cosh \mu l - 1) & \mu \sinh \mu l \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The determinant of he coefficient matrix is $2\mu(\cosh \mu l - 1)$ which vanishes only if $\mu = 0$ and so $a = b = 0$. If $\lambda = 0$, an obvious eigenfunction is $y = 1$. If $\lambda = \mu^2 > 0$, then

$$y(x) = a \sin \mu x + b \cos \mu x.$$

We see that

$$y(0) - y(l) = 0 \text{ if } b - (a \sin \mu l + b \cos \mu l) = 0$$

and

$$y'(0) - y'(l) = 0 \text{ if } a\mu - (a\mu \cos \mu l - b\mu \sin \mu l) = 0$$

or

$$\begin{pmatrix} -\sin \mu l & 1 - \cos \mu l \\ \mu(1 - \cos \mu l) & \mu \sin \mu l \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We obtain a nontrivial solution if

$$-\mu \sin^2 \mu l - \mu(1 - \cos \mu l)^2 = 0.$$

Expanding the second term and simplifying we arrive at $\cos\mu l = 1$ *and so*

$$\mu = \frac{2n\pi}{l}, \quad n = \pm 1, \pm 2, \cdots$$

In this case a, b *are arbitrary and so to each eigenvalue*

$$\lambda_n = \left(\frac{2n\pi}{l}\right) \quad n = \pm 1, \pm 2, \cdots$$

there are two linearly independent eigenfunctions

$$y_n(x) = a_n \sin\frac{2n\pi}{l}x + b_n \cos\frac{2n\pi}{l}x$$

5.3 Green's Functions

In this section we present an elementary introduction to the notion of a Green's function for the class of regular Sturm Liouville systems studied in the last section. In particular we interested in solving a RSLBVP when the differential equation has an extra nonhomogeneous right hand side. In order to simplify matters a bit, let us assume that the weight function in the previous sections is $p(x) \equiv 1$, so we consider the BVP$_\lambda$

$$L_\lambda(y) = (ky')' + g(x)y + \lambda y = 0, \qquad a < x < b$$

$$B_1(y) = 0$$
$$B_2(y) = 0$$

where

$$B_i(y) = \alpha_{i1}(a) + \alpha_{i2}y'(a) + \beta_{i3}y(b) + \beta_{i2}y'(b)$$

and $k \in C^1(a,b), k(x) > 0, x \in [a,b]$.

Definition 5.3.1. *A Green's function for BVP*$_\lambda$ *is a function* $G(x, \xi, \lambda)$ *for* $(x, \xi) \in [a,b] \times [a,b]$ *such that*

1. *The following hold*

   (a) $G(\cdot, \cdot, \lambda)$ *is continuous on* $[a,b] \times [a,b]$,

   (b) $\dfrac{\partial G}{\partial x}(\cdot, \xi, \lambda)$ *is continuous on* $[a, \xi) \times (\xi, b]$, *and,*

   (c) $\left.\dfrac{\partial G(x, \xi, \lambda)}{\partial x}\right|_{x=\xi^-}^{x=\xi^+} \equiv \dfrac{\partial G}{\partial x}(\xi^+, \xi, \lambda) - \dfrac{\partial G}{\partial x}(\xi^-, \xi, \lambda) = \dfrac{1}{k(\xi)}$

2. *for all* $\xi \in [a, b]$, $G(x, \xi, \lambda)$ *solves* $L_\lambda(G) = 0$, $x \neq \xi$.

3. *for all* $\xi \in (a, b)$, $B_i(G) = 0$.

Theorem 5.3.2. *If* λ *is not an eigenvalue of* BVP_λ*, then the boundary value problem has a unique Green's function* $G(x, \xi, \lambda)$ *and it is symmetric, i.e.,* $G(x, \xi, \lambda) = G(x, \lambda, \xi)$.

Proof. We provide a proof by construction for two special cases, (I.) Separated Boundary Conditions, (II.) Periodic Boundary Conditions (which are unseparated):

I.) Separated boundary conditions,

$$B_1(y) = \alpha_1 y(a) + \alpha_2 y'(a) = 0$$
$$B_2(y) = \beta_1 y(b) + \beta_2 y'(b) = 0.$$

Choose u_i such that $L_\lambda(u_i) = 0$ and $B_i(u_i) = 0$. This can be done as this amounts to solving an initial value problem corresponding to initial conditions specified at $x = a$ and $x = b$. The solutions u_1, u_2 must be linearly independent. Indeed, suppose $w = c_1 u_1 + c_2 u_2 \equiv 0$. Then

$$B_1(w) = c_2 B_1(u_2) = 0$$
$$B_2(w) = c_1 B_2(u_1) = 0.$$

If $B_1(u_2) = 0$ then u_2 would be an eigenfunction, but λ is not an eigenvalue. Hence we must have $c_2 = 0$. Similarly, $c_1 = 0$.

Now seek $G(x, \xi, \lambda)$ in the form

$$G(x, \xi, \lambda) = \begin{cases} Au_1(x) & a \leq x \leq \xi \\ Bu_2(x) & \xi \leq x \leq b. \end{cases}$$

We need

$$Au_1(\xi) = Bu_2(\xi)$$

and

$$Bu_2'(\xi) - Au_1'(\xi) = \frac{1}{k(\xi)}.$$

By Cramer's rule, one obtains

$$A = \frac{u_2(\xi)}{k(\xi)W(u_1, u_2)(\xi)}$$
$$B = \frac{u_1(\xi)}{k(\xi)W(u_1, u_2)(\xi)}$$

and hence

$$G(x,\xi,\lambda) = \begin{cases} \dfrac{u_1(x)u_2(\xi)}{k(\xi)W(u_1,u_2)(\xi)} & a \le x \le \xi \\ \dfrac{u_1(\xi)u_2(x)}{k(\xi)W(u_1,u_2)(\xi)} & \xi \le x \le b. \end{cases}$$

To see that G is symmetric, recall that Abel's formula states that

$$W(u_1,u_2)(\xi) = W(x_0)\exp\left(-\int_{x_0}^{\xi} \frac{k'(x)}{k(x)}dx\right) = W(x_0)\frac{k(x_0)}{k(\xi)}.$$

Hence

$$W(u_1,u_2)(\xi)k(\xi) = \text{constant}.$$

Example 5.3.3. *Find the Green's function for*

$$\begin{aligned} &y'' = 0, \;\; 0 < x < 1 \\ &y(0) = 0, \; y(1) = 0. \end{aligned}$$

Here $L_\lambda = L_0$ and it is easy to verify that $\lambda = 0$ is not an eigenvalue. Take $u_1(x) = x,\; u_2(x) = x - 1$ and

$$W(u_1,u_2) = \begin{vmatrix} x & x-1 \\ 1 & 1 \end{vmatrix} = 1$$

Hence

$$G(x,\xi,\lambda) = \begin{cases} x(\xi-1) & 0 \le x \le \xi \\ \xi(x-1) & \xi \le x \le 1. \end{cases}$$

Example 5.3.4. *Find the Green's function for*

$$\begin{aligned} &y'' + \lambda y = 0, \; \lambda > 0, \; 0 < x < \pi \\ &y(0) = 0, \; y(\pi) = 0. \end{aligned}$$

Here we regard $L(y) = y''$ and $L_\lambda(y) = y'' + \lambda y$. We know the eigenvalues are given by

$$\lambda_n = n^2, \; n = 1, 2, 3 \cdots$$

If $\lambda \ne n$, take $u_1(x) = \sin\sqrt{\lambda}x,\; u_2(x) = \sin\sqrt{\lambda}(x-\pi)x$. Then

$$k(0)W(u_1,u_2)(0) = \begin{vmatrix} 0 & -\sin\sqrt{\lambda}\pi \\ \sqrt{\lambda} & \sqrt{\lambda}\cos\sqrt{\lambda}\pi. \end{vmatrix} = -\sqrt{\lambda}\sin\sqrt{\lambda}\pi$$

and so

$$G(x,\xi,\lambda) = \begin{cases} -\dfrac{\sin\sqrt{\lambda}x \sin\sqrt{\lambda}(\xi-\pi)}{\sqrt{\lambda}\sin\sqrt{\lambda}\pi} & 0 \le x \le \xi \\ -\dfrac{\sin\sqrt{\lambda}\xi \sin\sqrt{\lambda}(x-\pi)}{\sqrt{\lambda}\sin\sqrt{\lambda}\pi} & \xi \le x \le 1. \end{cases}$$

II.) For our second example we consider the special case of Initial Value Problems. That is, in the special case in which the boundary conditions reduce to the initial values

$$B_1(u) = u(a)$$
$$B_2(u) = u'(a)$$

In this case we seek

$$G(x,\xi,\lambda) = \begin{cases} 0 & a \le x \le \xi \\ Au_1(x) + Bu_2(x) & \xi \le x \end{cases}$$

where u_1, u_2 are linearly independent solutions of $L_\lambda = 0$. In this case the continuity and jump condition give

$$Au_1(\xi) + Bu_2(\xi) = 0$$

and

$$Au_1'(\xi) + Bu_2'(\xi) = \frac{1}{k(\xi)}.$$

and so

$$A = -\frac{u_2(\xi)}{k(\xi)W(u_1,u_2)(\xi)}, \quad B = \frac{u_1(\xi)}{k(\xi)W(u_1,u_2)(\xi)}.$$

Hence

$$G(x,\xi,\lambda) = \begin{cases} 0 & a \le x \le \xi \\ \dfrac{u_1(\xi)u_2(x) - u_1(x)u_2(\xi)}{k(\xi)W(u_1,u_2)(\xi)} & \xi \le x. \end{cases}$$

Recall that the Heaviside function is defined by

$$H(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$$

Let $u_\xi(x)$ denote the solution of

$$L_\lambda(u_\xi(x)) = 0$$
$$u_\xi(\xi) = 0$$
$$u_\xi'(\xi) = \frac{1}{k(\xi)}.$$

Thus we see that the Green's function for the initial value problem satisfies

$$G(x,\xi) = H(x-\xi)u_\lambda(x).$$

Such a Green's function is often referred to as the causal fundamental solution. For more general boundary conditions, we might seek $G(x,\xi)$ in the form

$$G(x,\xi) = H(x-\xi)u_\xi + Au_1(x) + Bu_2(x)$$

where u_1, u_2 are linearly independent solutions of $L_\lambda = 0$.

Example 5.3.5. *Construct the Green's function for*

$$\begin{aligned} &u'' = 0, \quad 0 < x < 1 \\ &u(0) + u(1) = 0 \\ &u'(0) + u'(1) = 0 \end{aligned}$$

Seek

$$G(x,\xi) = H(x-\xi)u_\xi(x) + Ax + B \equiv E(x,\xi) + Ax + B$$

where

$$\begin{aligned} &u_\xi'' = 0, x > \xi > 0 \\ &u_\xi(\xi^+) = 0, u_\xi'(\xi^+) = 1. \end{aligned}$$

Then

$$E(x,\xi)) = \begin{cases} 0 & 0 \le x \le \xi \\ x-\xi & x \le \xi \le 1 \end{cases}$$

and

$$\begin{aligned} B_1(G) &= (E(0,\xi)+B) + (E(1,\xi)+A+B) \\ &= 2B + A + (1+\xi) = 0, \\ B_2(G) &= (0+A) + (1+A) \end{aligned}$$

Solving, one obtains

$$A = -1/2, \ B = -1/4 + \xi/2$$

and hence

$$G(x,\xi) = \begin{cases} \frac{1}{2}x - \frac{1}{4} + \frac{\xi}{2}, & 0 \le x \le \xi \\ (x-\xi) - \frac{1}{4} + \frac{\xi}{2} - \frac{x}{2}, & x < \xi \le 1 \end{cases}$$

or

$$G(x,\xi) = -\frac{1}{4} + \frac{|x-\xi|}{2}$$

□

We now turn to the main application of Green's function in this section. Namely, we consider the nonhomogeneous BVP.

$$L_\lambda(y) = (ky')' + g(x)y + \lambda y = f(x), \quad a < x < b$$

$$B_1(y) = 0$$
$$B_2(y) = 0$$

where

$$B_i(y) = \alpha_{i1}(a) + \alpha_{i2}y'(a) + \beta_{i3}y(b) + \beta_{i2}y'(b)$$

and $k \in C^1(a,b), k(x) > 0, x \in [a,b]$.

First we recall a classical formula whose general counterpart has far reaching consequences in the theory of ordinary and partial differential equations and the theory of weak solutions. At this point we will only consider a very special case. Namely, given any two functions u and v, a straightforward calculation gives the so-called Lagrange Identity:

$$vL_\lambda(u) - uL_\lambda(v) = \frac{d}{dx}P(u,v)$$

where (see (5.2.13))

$$P(u,v) = k\,(u'v - uv')$$

and we note that integration gives the Green's formula

$$\int_a^b [vL_\lambda(u) - uL_\lambda(v)] = P(u,v)|_{x=a}^{x=b}$$

Let $G(x,\xi)$ denote the Green's function for the homogeneous BVP$_\lambda$. From Lagrange's identity, for $x \neq \xi$

$$G(x,\xi)L_\lambda(y) - yL_\lambda(G(x,\xi)) = \frac{d}{dx}[k(Gy' - yG')]$$

which implies

$$\int_a^{\xi^-} GL_\lambda(y)dx = k(Gy' - G'y)]_a^{\xi^-}$$

and

$$\int_{\xi^+}^b GL_\lambda(y)dx = k(Gy' - G'y)]_{\xi^+}^b.$$

Hence

$$\int_a^b GL_\lambda(y)dx = k(Gy' - G'y)]_a^b - k(Gy' - G'y)]_{\xi^-}^{\xi^+}.$$

Suppose that B_1, B_2 are boundary conditions with the property that if u, v satisfy $B_1(u) = 0 = B_2(v)$, then

$$[k(Gy' - G'y)]_a^b = 0$$

Let us refer to such boundary conditions as regular boundary conditions. We have seen for example that separated boundary conditions are regular and regular boundary conditions result in a self-adjoint boundary value problem. Assume the the boundary conditions are regular and $B_(y) = B_2(y) = 0$. Then

$$\begin{aligned}\int_a^b GL_\lambda(y)dx &= -[k(Gy' - G'y)]_{\xi^-}^{\xi^+}\\ &= k[\frac{\partial G}{\partial x}(\xi^+, \xi) - \frac{\partial G}{\partial x}(\xi^-, \xi)]y(\xi)\\ &= y(\xi).\end{aligned}$$

Thus, formally at least, if y satisfies $L(y) = f$, then we should have $y(x) = \displaystyle\int_a^b G(x, \xi, \lambda) f(\xi)\, d\xi$. That is, again on a purely formal level, we have

$$\int_a^b G(x, \xi) L_\lambda(y)(x)dx = \int_a^b L_\lambda G(x, \xi)(y)(x)dx = y(\xi)$$

which suggest that $L_\lambda G(x, \xi) = \delta(x - \xi)$, i.e., the solution to

$$\begin{aligned}L_\lambda(y) &= f\\ B_i(y) &= 0\end{aligned}$$

would be given by

$$y(x) = \int_a^b G(x, \xi) f(\xi) d\xi$$

provided that λ is not an eigenvalue. This is indeed the case and we will argue this for separated boundary conditions.

Theorem 5.3.6. *If λ is not and eigenvalue of BVP_λ where the boundary conditions are separated, that is*

$$\begin{aligned}B_1(y) &= \alpha y(a) + \alpha y'(a)\\ B_2(y) &= \beta y(b) + \beta y'(b),\end{aligned}$$

then the unique solution of

$$L_\lambda(y) = f$$
$$B_1(y) = \gamma_1, B_2(y) = \gamma_2$$

is given by

$$y(x) = \frac{\gamma_2}{B_2(y_1)} y_1(x) + \frac{\gamma_1}{B_1(y_2)} y_2(x) + \int_a^b G(x,\xi) f(\xi) d\xi$$

where y_1, y_2 *are (not identically zero) solutions of*

$$L(y) = 0$$
$$B_1(y_1) = 0, B_2(y_2) = 0.$$

(Note that since y_1 *and* y_2 *are not identically zero, we must have* $B_1(y_2) \neq 0$ *and* $B_2(y_1) \neq 0$*.*

Proof. Since $B_1(G) = B_2(G) = 0$,

$$B_1(y) = \frac{\gamma_1}{B_1(y_2)} B_1(y_2) = \gamma_1$$

and similarly $B_2(y) = \gamma_2$

To see that the nonhomogeneous differential equation is satisfied, we consider

$$\begin{aligned} u(x) &= \int_a^b G(x,\xi) f(\xi) d\xi \\ &= \int_a^x G(x,\xi) f(\xi) d\xi + \int_x^b G(x,\xi) f(\xi) d\xi \end{aligned}$$

Then

$$\begin{aligned} u'(x) &= \int_a^{x^-} \tfrac{\partial G}{\partial x} f d\xi + G(x,x^-) f(x^-) \\ &+ \int_{x^+}^b \tfrac{\partial G}{\partial x} f d\xi - G(x,x^+) f(x^+) \\ &= \int_a^x G_x(x,\xi) f(\xi) d\xi + \int_x^b G_x(x,\xi) f(\xi) d\xi. \end{aligned}$$

Differentiating again we have

$$\begin{aligned} u''(x) &= \int_a^{x^-} G_{xx}(x,\xi) f(\xi) d\xi + G_x(x,x^-) f(x^-) \\ &+ \int_{x^+}^b G_{xx}(x,\xi) f(\xi) d\xi - G_x(x,x^+) f(x^+). \end{aligned}$$

We need the following observation,

$$\frac{\partial G}{\partial x}(x,x^-) = \frac{\partial G}{\partial x}(x^+,x)$$

$$\frac{\partial G}{\partial x}(x,x^+) = \frac{\partial G}{\partial x}(x^-,x).$$

For example, to verify the first of these claims, note that

$$\begin{aligned}\frac{\partial G}{\partial x}(x, x^-) &= \lim_{\epsilon\to 0^+} \frac{\partial G}{\partial x}(x, x-\epsilon)\\ &= \lim_{\epsilon\to 0^+} \lim_{h\to 0} \frac{G(x+h, x-\epsilon) - G(x, x-\epsilon)}{h}.\end{aligned}$$

The partials exist because we are in the open region $x > \xi$ away from the diagonal $(x = \xi)$ where only one-sided derivatives exist. Moreover, because G is smooth when $x > \xi$ we may interchange the order of limits to obtain

$$\begin{aligned}\frac{\partial G}{\partial x}(x, x^-) &= \lim_{\epsilon\to 0^+} \lim_{h\to 0^+} \frac{G(x+h, x-\epsilon) - G(x, x-\epsilon)}{h}\\ &= \lim_{h\to 0^+} \lim_{\epsilon\to 0^+} \frac{G(x+h, x-\epsilon) - G(x, x-\epsilon)}{h}\\ &= \lim_{h\to 0^+} \frac{G(x+h, x) - G(x, x)}{h}\\ &= \frac{\partial G}{\partial x}(x^+, x)\end{aligned}$$

Similarly for the other statement. Hence we have

$$\begin{aligned}u''(x) &= \int_a^b G_{xx}(x,\xi)f(\xi)d\xi + [G_x(x+, x) - G_x(x^-, x)]f(x^-)\\ &= \int_a^b G_{xx}(x,\xi)f(\xi)d\xi + f(x)/k(x).\end{aligned}$$

Thus

$$\begin{aligned}L_\lambda(u) &= ku'' + k'u' + gu + \lambda u\\ &= \int_a^b \Big[k(x)G_{xx}(x,\xi) + k'(x)G_x(x,\xi) + g(x)G(x,\xi)\\ &\qquad + \lambda G(x,\xi)\Big] f(\xi)d\xi + \frac{k(x)f(x)}{k(x)}\\ &= \int_a^b L_\lambda G(x,\xi)y(\xi)\,d\xi + f(x)\\ &= f(x)\end{aligned}$$

since $L_\lambda(G) = 0$. $\square$

Recall that we previously argued that the nonhomogeneous BVP

$$L(y) = f$$
$$B_1(y) = \gamma_1, B_2(y) = \gamma_2$$

has a unique solution if the BVP with the data $0, 0, 0$ has only the trivial solution. In light of the previous theorem we see that this is merely the statement that if $\lambda = 0$ is not an eigenvalue of BVP_λ, the nonhomogeneous BVP is uniquely solvable.

Theorem 5.3.6 is often phrased in the following equivalent form: Either the BVP

$$L_\lambda(u) = f$$
$$B_1(u) = \gamma_1, B_2(u) = \gamma_2$$

has a unique solution for each f or else the associated homogeneous problem has a nontrivial solution. This statement is referred to as the Fredholm Alternative.

Our results do not say that if λ is an eigenvalue the nonhomogeneous BVP is not solvable, and we address this situation now. First consider the analogy from linear algebra.

Remark 5.3.7. *Suppose A is an $n \times n$ matrix and we want to solve $Ax = b$. Then we have* ***Theorem:*** *(uniqueness) The solution to $Ax = b$ (if it exists) is unique if and only if $Ax = 0$ implies that $x = 0$.*

On the other hand the Fredholm Alternative gives an explicit criteria for existence. ***Theorem:*** *(existence) The equation $Ax = b$ has a solution if and only if $< b, v >= 0$ for every v satisfying $A^* v = 0$.*

Now suppose $A = A^T$ and let $M = A - \lambda I$. Then $M_y(\lambda) = f$ is solvable if and only if $f \perp N(M^T)$ (the nullspace of M^T). Since $M^T = M$ and $N(M) = \{v | Av = \lambda v\}$, we have $(A - \lambda I)y = f$ is solvable if and only if $y \perp v$, where $Av = \lambda v$. That is, if λ is an eigenvalue of A, then $(A - \lambda I)y = f$ has a solution if and only if f is orthogonal to the eigenspace corresponding to λ.

For the differential operator L_λ we have an analogous result. Again, we restrict our attention to separated boundary conditions.

Theorem 5.3.8. *Suppose that (μ, v) is an eigenpair for*

$$L_\lambda(y) = 0$$
$$B_i(y) = 0, \ i = 1, 2.$$

Then the nonhomogeneous problem

$$L_\mu(y) = f, \quad B_i(y) = 0, \quad i = 1, 2,$$

has a solution if and only if $\int_a^b f(x)v(x)dx = 0$.

Proof. Suppose u is a solution of the nonhomogeneous problem. Since u, v satisfy the homogeneous boundary conditions, we have

$$P(u,v) = k(uv' - u'v)\Big|_a^b = 0$$

Green's formula (5.2.13) gives

$$\int_a^b vf\,dx = \int_a^b [0 - vf]\,dx = \int_a^b [uL_\lambda(v) - vL_\lambda(u)]dx = 0.$$

Now suppose $\int_a^b vf\,dx = 0$ and let us choose u to the IVP solve $L_\lambda(u) = f$ with the initial conditions

$$u(a) = v(a),\ u'(a) = v'(a).$$

Then $B_1(u) = 0$ since $B_1(v) = 0$. We need to show that $B_2(u) = 0 = \beta_1 u(b) + \beta_2 u'(b)$. Green's formula gives

$$\int_a^b vf\,dx = \int_a^b [uL_\lambda(v) - vL_\lambda(u)]dx = k[uv' - vu']|_a^b.$$

But we already know that $B_1(u) = 0$ so this gives

$$0 = \int_a^b vf\,dx = k(b)[u(b)v'(b) - v(b)u'(b)].$$

Since $\beta_1^2 + \beta_2^2 \neq 0$, without loss of generality assume $\beta_1 \neq 0$. If $\beta_2 = 0$, then $\beta_1 v(b) = 0$ so $v(b) = 0$ and hence $u(b) = 0$ and so $B_2(u) = 0$. If $\beta_2 \neq 0$, then $\beta_1/\beta_2 v(b) = v'(b)$. Since v is a nontrivial solution of $L_\lambda(y) = 0$, we see from the previous relations that $v(b) \neq 0$ and $v'(b) \neq 0$. Thus is follows that

$$u(b) = \frac{v(b)u'(b)}{v'(b)}$$

and so

$$\beta_1 u(b) + \beta_2 u'(b) = \beta_1 \frac{v(b)}{v'(b)} u'(b) + \beta_2 u'(b)$$

$$= u'(b)\left(\beta_1 \frac{v(b)}{v'(b)} + \beta_2\right) = 0$$

and hence $B_2(u) = 0$. Thus u solves the nonhomogeneous boundary value problem. □

Exercises for Chapter 5

1. Consider
$$k_0 y'' + g(x)y = 0 \qquad (*)$$
where
$$k_0 > 0, \;\; 0 < g_1 < g(x) < g_2.$$
If z_1, z_2 are consecutive zeros of a solution to (*), show that
$$\pi\sqrt{\frac{k_0}{g_2}} \le (z_2 - z_1) \le \pi\sqrt{\frac{k_0}{g_1}}.$$

2. Suppose that $y(x)$ is a solution of
$$y'' + a(x)y = 0.$$
   (a) If $a(x) \ge m > 0,\; x_0 \le x \le \infty$, show that $y(x)$ has infinitely many zeros but if $a(x) < 0,\; x_0 \le x \le \infty$, then $y(x)$ has at most one zero.

   (b) If
$$\lim_{x\to\infty} a(x) = a_0 > 0,$$
   prove that the distance between consecutive zeros tends to $\pi/\sqrt{a_0}$ as the zeros tend to infinity.

3. Find the eigenvalues and eigenfunctions of $u'' + \lambda u = 0$ with the boundary conditions:
   (a) $u(0) = u(1) = 0$,
   (b) $u'(0) = u'(1) = 0$,
   (c) $u(0) = 0, u(1) - u'(1) = 0$.

4. Find the Green's function for
$$y'' - \gamma^2 y = 0, \;\; y'(0) = 0, \;\; y(1) = 0, \;\; \gamma > 0.$$

5. Find the eigenvalues and eigenfunctions for the following boundary value problem and then construct the Green's function.
$$\frac{d}{dx}\left(x\frac{dy}{dx}\right) + \frac{\lambda}{x}y = 0, \;\; 1 < x < e$$
$$u(1) = 0, \; u(e) = 0.$$

6. Do parts a) through c):

   (a) Show that the eigenvalues and eigenfunctions of the boundary value problem

$$\frac{d}{dx}\left((1+x)^2\frac{d}{dx}(u)\right)+\lambda u=0,\quad 0<x<1$$

$$u(0)=u(1)=0$$

   are given by

$$\lambda_n=\left(\frac{n\pi}{\ln 2}\right)^2+\frac{1}{4},\quad n=1,2,3,\ldots$$

$$u_n(x)=\frac{\sin\left((n\pi\ln(1+x))/(\ln 2)\right)}{\sqrt{1+x}}.$$

   (b) Construct the Green's function for this problem.

   (c) If

$$f(x)=\frac{1}{\sqrt{1+x}},\quad \lambda=\frac{1}{4}+\left(\frac{\pi}{\ln 2}\right)^2,$$

   does

$$L(u)+\lambda u=f(x)$$

$$u(0)=u(1)=0$$

   have a solution? If so, is it unique?

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