Chapter 3

Linear equations

3.1 Introduction

We will study linear systems of the form

\[ y'(t) = A(t)y(t) + B(t) \]  \hspace{1cm} (3.1.1)

where \( B(t), y(t) \in \mathbb{R}^n \), \( A \) is \( n \times n \). Recall, for instance, that we can always rewrite an \( n \)th order linear equation

\[ z^{(n)} + a_1(t)z^{(n-1)}(t) + \cdots + a_n(t)z(t) = b(t) \]  \hspace{1cm} (3.1.2)

in the form of (3.1.1). Indeed if we define

\[
    y = \begin{pmatrix}
        y_1 \\
        y_2 \\
        \vdots \\
        y_n
    \end{pmatrix} = \begin{pmatrix}
        z \\
        z^{(1)} \\
        \vdots \\
        z^{(n-1)}
    \end{pmatrix}
\]

then

\[
y'(t) \equiv \begin{pmatrix}
    0 & 1 & 0 & \cdots & 0 \\
    0 & 0 & 1 & \cdots & 0 \\
    \vdots & \vdots & \ddots & \ddots & \vdots \\
    -a_n(t) & -a_{n-1}(t) & \cdots & -a_1(t)
\end{pmatrix} y + \begin{pmatrix}
    0 \\
    0 \\
    \vdots \\
    b(t)
\end{pmatrix} = A(t)y + B(t)
\]
Recall that the matrix norm is defined by
\[ \|A\| = \sup_{|y| = 1} |Ay| = \sup_{y \neq 0} \frac{|Ay|}{|y|}. \]
Here we use the sup norm, \(|y| = |y|_\infty = \max_i \{|y_i|\} \), in which case
\[ \|A\| = \max_i \sum_k |a_{ik}|. \]

**Theorem 3.1.1.** If \( A(t), B(t) \) are continuous on \([a, b]\), \( t_0 \in (a, b) \), then the IVP
\[
y' = A(t)y + B(t) \\
y(t_0) = y_0
\]
has a unique solution that exists on \((a, b)\).

*Proof.* In the notation of the Fundamental Existence Theorem,
\[
f(t, y) = A(t)y + B(t)
\]
and so \( f \) is continuous on \((a, b) \times \mathbb{R}^n\) and \( f \) is locally Lipschitz in the second variable. To verify this last claim note that the partials
\[
\frac{\partial f_i}{\partial y_j}(t) = a_{ij}(t)
\]
are continuous or alternatively, observe that \(|f(t, w) - f(t, u)| \leq \|A\| \|w - u\|\).

To show the solution exists for all \( t \) we need only show it is finite for all \( t \in (a, b) \). If \( t > t_0 \)
\[
y(t) = y_0 + \int_{t_0}^{t} (A(s)y(s) + B(s))ds
\]
\[
\Rightarrow |y(t)| \leq |y_0| + \int_{t_0}^{t} (\|A(s)\| |y(s)| + |B(s)|)ds
\]
\[
\leq |y_0| + \max_{s \in [a, b]} |B(s)|(b - a) + \max_{s \in [a, b]} \|A(s)\| \int_{t_0}^{t} |y(s)|ds
\]
or
\[
|y(t)| \leq K_1 + K_2 \int_{t_0}^{t} |y(s)|ds.
\]
3.1. INTRODUCTION

Hence by Gronwall,

\[ |y(t)| \leq K_1 e^{K_2(t-t_0)}, \quad t_0 \leq t < b. \]

If \( t < t_0 \), then let \( t = \eta(s) = (a - t_0)s + t_0 \) so that as \( s \) varies from 0 to 1, \( t \) varies from \( t_0 \) down to \( a \). We let \( z(s) = y(\eta(s)) = y(t) \) and we can write

\[ z(s) = y(t) = y_0 + \int_{t_0}^{(a-t_0)s+t_0} (A(\tau)y(\tau) + B(\tau)) \, d\tau \]

( Let \( \tau = (a-t_0)u + t_0 \) )

\[ y_0 + \int_0^s (A(\eta(u))y(\eta(u)) + B(\eta(u))) (a-t_0) \, du \]

\[ = y_0 + (a-t_0) \int_0^s (A(\eta(u))z(u) + B(\eta(u))) \, du. \]

Thus with \( K_1 = |y_0| + \max_{s\in[a,b]} |B(s)|(b-a) \) and \( K_2 = \max_{s\in[a,b]} \|A(s)\| \) we have

\[ |z(s)| \leq K_1 + |(t_0-a)|K_2 \int_0^s |z(s)| \, ds. \]

Hence by Gronwall,

\[ |z(s)| \leq K_1 e^{(t_0-a)|K_2(s-0)}, \quad 0 \leq s < 1. \]

So that

\[ |y(t)| \leq K_1 e^{(t_0-a)|K_2((t-t_0)/(a-t_0))}, \quad a \leq t \leq t_0. \]

Thus finally we have

\[ |y(t)| \leq K_1 e^{K_2|t-t_0|}, \quad t \in [a,b]. \]

The following corollary is an obvious but important consequence of the preceding theorem.

**Corollary 3.1.2.** If \( a_i(t) \in C^0[a,b], a_n(t) \neq 0 \), then the IVP

\[ y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_1(t)y = 0 \]

\[ y(t_0) = \gamma_1, \ldots, y^{(n-1)}(t_0) = \gamma_n \]

has a unique solution that exists on \( (a,b) \).
CHAPTER 3. LINEAR EQUATIONS

The above arguments show that \( y(t) \) is defined on \((a, b)\) if \( A(t), B(t) \) are continuous on \([a, b]\). In fact \( y(t) \) can be extended to \([a, b]\). First observe that \( y(a^+), y(b^-) \) exist. Indeed, let \( t_n \to b \) and note

\[
|y(t_m) - y(t_n)| \leq \int_{t_n}^{t_m} \left( |A(s)y(s) + B(s)| \right) ds \\
\leq \int_{t_n}^{t_m} \left( \|A(s)\| |y(s)| + |B(s)| \right) ds.
\]

From the above arguments we see that

\[
|y(t)| \leq K_1 e^{K_2(b-a)} = C.
\]

Hence

\[
|y(t_m) - y(t_n)| \leq (\max_{[a,b]} \|A(s)\| C + \max_{[a,b]} |B(s)|)(t_m - t_n).
\]

Thus \( \{y(t_n)\} \) is Cauchy and so the limit exists. Since \( y'(t) = A(t)y(t) + B(t) \) for all \( t \in (a, b) \) we get

\[
\lim_{t \to b^-} y'(t) = A(b)y(b^-) + B(b).
\]

Hence the differential equation is satisfied on \((a, b)\), and in a similar way one argues that the equation is satisfied in \([a, b]\).

3.2 \( N^{th} \) order linear equations

We will consider (3.1.2) assuming \( a_i(t), b_i(t) \in C[a, b] \) and \( a_n(t) \neq 0 \) for \( t \in [a, b] \). Thus we consider

\[
y^{(n)} + a_1(t)y^{(n-2)} + a_2(t)y^{(n-2)} + \cdots + a_n(t)y = \beta(t) \quad (3.2.1)
\]

which gives

\[
\begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-a_n & -a_{n-1} & \cdots & a_1 \\
\end{pmatrix}
\]

Let

\[
L(\cdot) = \frac{d^n(\cdot)}{dt^n} + a_1 \frac{d^{n-1}(\cdot)}{dt^{n-1}} + \cdots + a_n(\cdot).
\]
Then (3.2.1) may be written as the linear nonhomogeneous system

\[ L(y) = \beta. \]  

(3.2.2)

The operator \( L \) is said to be linear since

\[ L(cy_1(t) + c_2y_2(t)) = c_1L(y_1(t)) + c_2L(y_2(t)). \]

It easily follows that the set of solutions of the homogeneous linear system

\[ L(y) = 0 \]  

(3.2.3)

is a vector space.

The Wronskian of \( \{y_j(t)\}_{j=1}^n \) is defined as

\[ W(t) \equiv W(\{y_j(t)\}_{j=1}^n)(t) = W(t) = \\
\begin{vmatrix}
y_1(t) & \cdots & y_n(t) \\
\vdots & \vdots & \vdots \\
y_1^{(n-1)}(t) & \cdots & y_n^{(n-1)}(t)
\end{vmatrix}. \]

**Theorem 3.2.1.** [Abel’s Formula] If \( y_1, \ldots, y_n \) are solutions of (LH), and \( t_0 \in (a, b) \), then

\[ W(t) = W(t_0) \exp \left[ - \int_{t_0}^t \alpha_1(s) \, ds \right]. \]

Thus the Wronskian of \( \{y_1, \ldots, y_n\} \) is never 0 or identically 0.

**Proof.** We compute

\[
W'(t) = \\
\begin{vmatrix}
y_1' & \cdots & y_n' \\
y_1' & \cdots & y_n' \\
\vdots & \vdots & \vdots \\
y_1^{(n-1)} & \cdots & y_1^{(n-1)}
\end{vmatrix}
+ \begin{vmatrix}
y_1 & \cdots & y_n \\
y_1' & \cdots & y_n' \\
\vdots & \vdots & \vdots \\
y_1^{(n-1)} & \cdots & y_n^{(n-1)}
\end{vmatrix}
+ \cdots + \\
\begin{vmatrix}
y_1 & \cdots & y_n \\
y_1' & \cdots & y_n' \\
\vdots & \vdots & \vdots \\
y_1^{(n-1)} & \cdots & y_n^{(n-1)}
\end{vmatrix}
= \\
\begin{vmatrix}
y_1 & \cdots & y_n \\
y_1' & \cdots & y_n' \\
\vdots & \vdots & \vdots \\
y_1^{(n-2)} & \cdots & y_n^{(n-2)}
\end{vmatrix}
- \sum_{j=1}^n \alpha_j y_1^{(n-j)} \cdots - \sum_{j=1}^n \alpha_j y_n^{(n-j)}
\]
\[ \begin{vmatrix} y_1 & \cdots & y_n \\ y'_1 & \cdots & y'_n \\ \vdots & \vdots & \vdots \\ -\alpha_1(t)y_1^{(n-1)} & \cdots & -\alpha_1(t)y_n^{(n-1)} \end{vmatrix} = -\alpha_1(t)W(t). \]

Hence
\[ W(t) = Ke^{-\int_{t_0}^{t} \alpha_1(ts)ds} \]
or, more explicitly,
\[ W(t) = W(t_0)e^{-\int_{0}^{t} \alpha_1(s)ds}. \]

\[ \square \]

**Definition 3.2.2.** A collection of functions \(\{y_i(t)\}_{i=1}^{k}\) is linearly independent on \((a, b)\) if
\[ \sum_{i=1}^{k} c_i y_i(t) = 0 \quad \text{for all} \quad x \in (a, b) \Rightarrow c_j = 0 \quad \text{for} \quad j = 1, \cdots, n. \]

Otherwise we say the set \(\{y_i(t)\}\) is linearly dependent.

**Theorem 3.2.3.** Suppose \(y_1, \ldots, y_n\) are solutions of \((LH)\). If the functions are linearly dependent on \((a, b)\) then \(W(t) = 0\) for all \(t \in (a, b)\). Conversely, if there is an \(t_0 \in (a, b)\) so that \(W(t_0) = 0\), then \(W(t) = 0\) for all \(t \in (a, b)\) and the \(y_i(t)\) are linearly dependent on \((a, b)\).

**Proof.** If the \(\{y_i(t)\}\) are linearly dependent, then there are \(c_i(t)\) not all zero such that
\[ \sum_{i} c_i y_i(t) = 0 \quad \text{for all} \quad t \in (a, b) \]
\[ \Rightarrow \sum_{i} c_i y_i^{(k)}(t) = 0 \quad \text{for all} \quad t \quad \text{any} \quad k. \]

Hence, defining
\[ M(t) = \begin{bmatrix} y_1(t_0) & \cdots & y_n(t_0) \\ \vdots & \ddots & \vdots \\ y_1^{(n-1)}(t_0) & \cdots & y_n^{(n-1)}(t_0) \end{bmatrix}, \quad C = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}, \]
the system can be written as
\[ M(t)C = 0 \]
and since \(C \neq 0\) we see that \(M\) is singular and therefore
\[ W(t) = \det(M(t))0 \quad \text{for all} \quad x \in (a, b). \]
Conversely, if det$(M(t)) = W(t_0) = 0$ then
\[ M(t)C = 0 \]
has a nontrivial solution. For this choice of $c_i$'s, let
\[ y(t) = \sum c_i y_i(t). \]
Then
\[ y(t_0) = 0, \quad y'(t_0) = 0, \quad \ldots, \quad y^{(n-1)}(t_0) = 0. \]
and since $y$ is a solution of $Ly = 0$, from the uniqueness part of the fundamental existence uniqueness theorem, we must have $y(t) = 0$ for all $x \in (a, b)$. \hfill \Box

**Example 3.2.4.** Consider $y_1 = t^2$, $y_2 = t|t|$. Then
\[ W(t) = \begin{vmatrix} t^2 & t|t| \\ 2x & 2x \ sgn(t) \end{vmatrix} = 2t^3 \ sgn(t) - 2t^2|t| \equiv 0. \]
However, $y_1(t)$, $y_2(t)$ are not linearly dependent. For suppose
\[ c_1 y_1(t) + c_2 y_2(t) = 0, \quad \text{for all } t. \]
Then
\[ c_1 t + c_2 |t| = 0 \quad \text{for all } t. \]
If $t > 0$,
\[ c_1 t + c_2 t = 0 \Rightarrow c_1 = -c_2 \]
while for $t < 0$,
\[ c_1 t - c_2 t = 0 \Rightarrow c_1 = c_2. \]
Hence $c_1 = c_2 = 0$ and so $y_1$, $y_2$ are linearly independent on any interval $(a, b)$ containing 0. Thus $y_1$, $y_2$ are not solutions of a linear homogeneous 2nd order equation on $(a, b)$.

**Theorem 3.2.5.** 1. There are $n$ linearly independent solutions of (3.2.3)

2. If $\{y_1, \ldots, y_n\}$ are linearly independent solutions of (3.2.3), then given any solution $y$ of (3.2.3), there exists unique constants $c_1, \ldots, c_n$ such that
\[ y(t) = c_1 y_1 + \cdots + c_n y_n. \]
CHAPTER 3. LINEAR EQUATIONS

Proof. For part 1., we denote by \( \{e_j\}_{j=1}^n \) the usual unit basis vectors in \( \mathbb{R}^n \) where \( e_j \) is 1 in the \( j \)th position and 0 otherwise. Then let \( y_j \) denote the solution of the initial value problem \( Ly = 0 \) with

\[
\begin{bmatrix}
y_j(t_0) \\
y_j^{(1)}(t_0) \\
\vdots \\
y_j^{(n-1)}(t_0)
\end{bmatrix} = e_j,
\]

so that

\[ W(t_0) = \det(I_n) = 1, \quad (I_n \text{ the } n \times n \text{ identity matrix} \]

and so the solutions are linearly independent.

For part 2., let \( y \) be a solution of \( L(y) = 0 \) and let \( \{y_j\}_{j=1}^n \) be linearly independent solutions. Consider the solution

\[ \varphi(t) = c_1 y_1(t) + \cdots + c_n y_n(t) \]

where \( c_1, \ldots, c_n \) are arbitrary constants. We seek \( c_1, \ldots, c_n \) so that

\[
\begin{align*}
\varphi(t_0) &= c_1 y_1(t_0) + \cdots + c_n y_n(t_0) = y(t_0) \\
\vdots \\
\varphi^{(n-1)}(t_0) &= c_1 y_1^{(n-1)}(t_0) + \cdots + c_n y_n^{(n-1)}(t_0) = y^{(n-1)}(t_0)
\end{align*}
\]

or

\[
\begin{pmatrix}
y_1(t_0) & \cdots & y_n(t_0) \\
\vdots & \vdots & \vdots \\
y_1^{(n-1)}(t_0) & \cdots & y_n^{(n-1)}(t_0)
\end{pmatrix}
\begin{pmatrix}
c_1 \\
c_n
\end{pmatrix}
= 
\begin{pmatrix}
y(t_0) \\
\vdots \\
y^{(n-1)}(t_0)
\end{pmatrix}
\]

This system has a unique solution since \( W(t_0) \neq 0 \). Hence by uniqueness

\[ \varphi(t) = y(t) \text{ for all } t. \]

\[ \square \]

**Definition 3.2.6.** A set of \( n \) linearly independent solutions is called a fundamental set. The above result shows that a fundamental set is a basis for the vector space of all solutions of (3.2.2). If \( \{y_1, \ldots, y_n\} \) is a fundamental set we call

\[ c_1 y_1(t) + \cdots + c_n y_n(t) \]

a general solution of (3.2.2). A general solution is a complete solution.
Theorem 3.2.7. If $y_p$ is any particular solution of $Ly = \beta$, then any other solution of $Ly = \beta$ can be expressed as
$$y(t) = y_p(t) + y_h(t)$$
where $y_h(t)$ is a solution of the homogeneous problem $Ly = 0$.

Proof. Since $L(y - y_p) = 0$, we can write
$$y - y_p = c_1y_1 + \cdots + c ny_n = y_h$$
where $\{y_1, \cdots, y_n\}$ is a fundamental set.

We call $y_p(t)$ a particular solution of (3.2.2) and $y_p + y_h$ the general solution of (3.2.2).

The problem of finding a basis of solutions of the homogeneous problem is, in general, very difficult. Indeed a considerable effort was directed at constructing solutions of specific equations with non constant coefficients. Many of the equations and solutions now bare the names of those that first worked out the details. Most of these solutions were obtained from a theory that we will not discuss in this class – the method of power series. The basic idea is that if an operator $L = a_0D^n + a_1D^{(n-1)} + \cdots + a_n$ has analytic coefficients, then the solutions of $Ly = 0$ will be analytic functions. So, if $a_0(t) \neq 0$, we seek solutions in the form $y = \sum_{k=0}^{\infty} c_k t^k$. Substituting this expression into the equation we attempt to determine the coefficients $\{c_k\}_{k=0}^{\infty}$. Most importantly is the case in which $a_0$ is zero at some point (called a singular point). In the special case of a so called regular singular point we can apply the method of Frobenius to seek solutions in the form $y = t^r \sum_{k=0}^{\infty} c_k t^k$ where $r$ is obtained from the so-called indicial equation.

We will not pursue this any further at this point. Rather we now turn to a classical method that can be used to reduce the order of an equation if one solution can be found by some method (like guessing for example).

Reduction of Order:

Consider
$$Ly = y^{(n)} + a_1y^{(n-1)} + \cdots + a_n y = 0$$
under the assumption that we have a solution $y_1$, i.e., $L(y_1) = 0$. We seek a second solution in the form $y = uy_1$. To find $u$ we substitute the $y$ into the equation to obtain.
$$(uy_1)^{(n)} + a_1(uy_1)^{(n-1)} + \cdots + a_n(uy) = 0.$$
CHAPTER 3. LINEAR EQUATIONS

Now we apply Leibniz formula

\[(f \cdot g)^{(n)} = \sum_{j=0}^{n} \binom{n}{j} f^{(n-j)} g^{(j)}\]

to obtain

\[\sum_{j=0}^{n} \binom{n}{j} u^{(n-j)} y_1^{(j)} + a_1 \sum_{j=0}^{n-1} \binom{n-1}{j} u^{(n-j)} y_1^{(j)} + \cdots + a_n u y_1 = 0.\]

The coefficient of the terms involving \(u\) but no derivatives of \(u\) are \(uL(y_1)\) which is zero. Thus if we let \(v = u'\) we obtain an equation for \(v\) of order \((n - 1)\) in the form

\[y_1 v^{(n-1)} + \cdots \left[ ny_1^{(n-1)} + a_1 (n-1) y_1^{(n-2)} + \cdots + a_{n-1} y_1 \right] v = 0.\]

Provided that \(y_1(t) \neq 0\) we know that there exists a fundamental set of solutions to this problem, \(\{v_j\}_{j=2}^{n}\). With this set, a fundamental set of solutions to the original problem is given in terms of \(u_j = \int v_j(s) \, ds\) as

\[y_1, y_1 u_2, \ldots, y_1 u_n.\]

For the special case \(Ly = y'' + a_1 y' + a_2 y = 0\), if \(y_1\) is a solution we seek a second solution \(y = uy_1\) which as above, substituting \(y\) into the equation, leads to the equation of order one for \(v = u'\)

\[y_1 v' + (2y_1' + a_1 y_1) v = 0.\]

If \(y_1 \neq 0\), we multiply by \(y_1\) to obtain

\[(y_1^2 v)' + a_1 (y_1^2 v) = 0,\]

which implies

\[y_1^2(t) v(t) = c \exp \left[ - \int_{t_0}^{t} a_1(s) \, ds \right].\]

We set \(c = 1\) to obtain \(v(t)\)

\[v(t) = \frac{1}{y_1(t)^2} \exp \left[ - \int_{t_0}^{x} a_1(s) \, ds \right].\]

Now we can find \(u\) and thus obtain a second solution \(y_2\) to \(Ly = 0\) given by

\[y_2(t) = y_1(t) \int_{t_0}^{t} \frac{1}{y_1(s)^2} \exp \left[ - \int_{t_0}^{s} a_1(\xi) \, d\xi \right] \, ds.\]

We now introduce two methods for constructing a particular solution of the nonhomogeneous equation when \(n = 2\).
Variation of Parameters for Nonhomogeneous Equations:

Consider the second order case of the equation $L(y) = \beta$ and suppose \{y_1, y_2\} is a fundamental set. Then $c_1y_1(t) + c_2y_2(t)$ is a general solution of $L(y) = 0$. A method due to Lagrange for solving $L(y) = \beta$ is based on the idea of seeking a solution as

$$y_p(t) = c_1(t)y_1(t) + c_2(t)y_2(t).$$

Then

$$y_p' = c_1y_1' + c_2y_2' + c_1'y_1 + c_2'y_2.$$

To simplify the algebra, we impose the auxiliary condition

$$c_1'y_1 + c_2'y_2 = 0.$$

Then

$$y_p'' = c_1y_1'' + c_2y_2'' + c_1'y_1' + c_2'y_2'.$$

If we substitute into $L(y) = \beta$, we want

$$c_1(t)(y_1'' + a_1y_1' + a_0y_1) + c_2(t)(y_2'' + a_1y_2' + a_0y_2) + c_1'y_1' + c_2'y_2' = \beta(t).$$

Note that the two parenthesized expressions are zero because $y_1$ and $y_2$ are solutions of the homogeneous equation. Thus we need to solve

$$c_1'y_1 + c_2'y_2 = 0$$

$$c_1'y_1' + c_2'y_2' = \beta.$$

By Cramer’s Rule

$$c_1'(t) = \frac{-y_2(t)\beta_1(t)}{W(y_1, y_2)(t)}, \quad c_2'(t) = \frac{y_1(t)\beta(t)}{W(y_1, y_2)(t)}.$$

Thus a particular solution is given as

$$y_p(t) = -y_1(t)\int_{t_0}^{t} \frac{y_2(s)\beta(s)}{W(s)} \, ds + y_2(t)\int_{t_0}^{t} \frac{y_1(s)\beta(s)}{W(s)} \, ds$$

$$= \int_{t_0}^{t} \left[ \frac{y_1(s)y_2(s) - y_1(s)y_2(s)}{W(y_1, y_2)(s)} \right] \beta(s) \, ds$$

$$= \int_{t_0}^{t} g(x, s)\beta(t) \, ds.$$

$g(t, s)$ is called a Fundamental solution.
The same method works, if not as smoothly, in the general case. Consider the equation $L(y) = \beta$ where $L$ has order $n$, and let $\{y_1, \ldots, y_n\}$ be a fundamental set of solutions of the homogeneous problem $L(y) = 0$. As in the last section, given a basis of solutions $\{y_j\}_{j=1}^n$ of the homogeneous problem $Ly = 0$ we seek a solution of $L(y) = \beta$ in the form

$$y_p(t) = u_1(t)y_1(t) + \cdots + u_n(t)y_n(t).$$

We seek a system of equations that can be solved to find $u'_1, \ldots, u'_n$. To this end we note that by applying the product rule to $y_p$ and, collecting terms carefully, we can conclude that

$$u'_1y_1 + u'_2y_2 + \cdots + u'_ny_n = 0 \quad \Rightarrow \quad y'_p = u_1y'_1 + u_2y'_2 + \cdots + u_ny'_n$$

$$u'_1y'_1 + u'_2y'_2 + \cdots + u'_ny'_n = 0 \quad \Rightarrow \quad y''_p = u_1y''_1 + u_2y''_2 + \cdots + u_ny''_n$$

$$u'_1y''_1 + u'_2y''_2 + \cdots + u'_ny''_n = 0 \quad \Rightarrow \quad y'''_p = u_1y'''_1 + u_2y'''_2 + \cdots + u_ny'''_n$$

$$\vdots \quad \Rightarrow \quad \vdots$$

$$u'_1y^{(n-2)}_1 + u'_2y^{(n-2)}_2 + \cdots + u'_ny^{(n-2)}_n = 0 \quad \Rightarrow \quad y^{(n-1)}_p = u_1y^{(n-1)}_1 + u_2y^{(n-1)}_2 + \cdots + u_ny^{(n-1)}_n$$

$$u'_1y^{(n-1)}_1 + u'_2y^{(n-1)}_2 + \cdots + u'_ny^{(n-1)}_n = \beta \quad \Rightarrow \quad y^{(n)}_p = u_1y^{(n)}_1 + u_2y^{(n)}_2 + \cdots + u_ny^{(n)}_n + \beta$$

which implies

$$L(y_p) = u_1L(y_1) + u_2L(y_2) + \cdots + u_nL(y_n) + \beta = \beta.$$  

Now we note that the system of equations becomes

$$
\begin{bmatrix}
y_1 & y_2 & \cdots & y_n \\
y'_1 & y'_2 & \cdots & y'_n \\
\vdots & \vdots & \cdots & \vdots \\
y^{(n-1)}_1 & y^{(n-1)}_2 & \cdots & y^{(n-1)}_n
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2 \\
\vdots \\
u_n
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
\vdots \\
\beta
\end{bmatrix}.
$$

The determinant of the coefficient matrix is nonvanishing since it is the Wronskian $W(t)$ of a set of linearly independent solutions to an $n$-order linear differential equation. Applying Kramer’s rule we can write the solutions as

$$u'_k(t) = \frac{W_k(t)}{W(t)}, \quad k = 1, \ldots, n$$
where $W_k(t)$ is the determinant of the matrix obtained from the coefficient matrix by replacing the $k$th column $\begin{bmatrix} y_k & y_k' & \cdots & y_k^{(n-1)} \end{bmatrix}^T$ by the vector $[0 \ 0 \ \cdots \ \beta]^T$.

If we define

$$g(t, s) = \sum_{k=1}^{n} \frac{y_k(t)W_k(s)}{W(s)}$$

then a particular solution of $L(y) = \beta$ is

$$y_p = \int_{t_0}^{t} g(t, s) \beta(s) \, ds.$$  

**Linear Constant Coefficients Equations Revisited:**

We have already learned, in Chapter 1, how to find a set of $n$ solutions to any homogeneous equation of the form $Ly = 0$ with $L = D^{(n)} + a_1D^{(n-1)} + \cdots + a_{n-1}D + a_n$. Namely, we factor the operator into a product of factors $(D - r)^k$ and $(D^2 - 2\alpha D + \alpha^2 + \beta^2)^m$. Having done this we simply observe that the general solution of the associated homogeneous problem for each of these types of operators is easy to write out. Namely, we have

$$(D - r)^k y = 0 \quad \Rightarrow \quad y = \sum_{j=1}^{k} c_j t^{j-1} e^{rt} \quad (3.2.4)$$

$$(D^2 - 2\alpha D + \alpha^2 + \beta^2)^m y = 0 \quad \Rightarrow \quad y = \sum_{j=1}^{k} c_j t^{j-1} e^{\alpha t} \cos(\beta t)$$

$$+ \sum_{j=1}^{k} c_j t^{j-1} e^{\alpha t} \sin(\beta t) \quad (3.2.5)$$

In the case that the coefficients $a_i$ are constant, it is possible to describe the solutions explicitly by simply solving the homogeneous equation for each factor and adding these terms together. What we have not proved is that all such solutions give a basis for the null space of $L$, i.e., we have not shown that the solutions are linearly independent. To show that these solutions are linearly independent is not really difficult but to do it completely rigorously and carefully is a bit lengthy.

First we note that

**Lemma 3.2.8.** If $\lambda = \alpha + \beta i$ is a real ($\beta = 0$) or complex number, then

$$y = \left( \sum_{j=1}^{k} c_j t^{j-1} \right) e^{\lambda t}$$
is the complete solution of \((D - \lambda)^k y = 0\).

**Proof.** Showing that the solutions of \((D - \lambda)^k y = 0\) are linearly independent amounts to showing that

\[
\left( \sum_{j=1}^{k} c_j t^{j-1} \right) e^{\lambda t} = 0 \quad \text{for all } t \in \mathbb{R} \Rightarrow c_j = 0, \; j = 1, 2, \ldots, k.
\]

But, on noting that \(e^{\lambda t} \neq 0\) and dividing, this result is obvious from the fundamental theorem of algebra which says that a polynomial of degree \(k\) has exactly \(k\) zeros. \(\square\)

**Lemma 3.2.9.** If \(\lambda_1 \neq \lambda_2\) are two complex numbers and

\[
p(t) = \sum_{j=1}^{k} c_j t^{j-1} \quad \text{and} \quad q(t) = \sum_{j=1}^{\ell} d_j t^{j-1},
\]

are two polynomials, then

\[
p(t) e^{\lambda_1 t} = q(t) e^{\lambda_2 t} \quad \text{for all } t \in \mathbb{R} \Rightarrow p(t) = 0, \quad q(t) = 0.
\]

**Proof.** To see that this is true we first multiply both sides of the equation by \(e^{-\lambda_1 t}\) so that

\[
p(t) = q(t) e^{(\lambda_2 - \lambda_1) t} \quad \text{for all } t \in \mathbb{R}.
\]

Now consider the cases in which \(\alpha < 0, \alpha > 0\) and \(\alpha = 0\) where \((\lambda_2 - \lambda_1) \equiv \alpha + \beta i\). If \(\alpha < 0\) then (using L’Hospital’s rule in the first term)

\[
\lim_{t \to +\infty} q(t) e^{(\lambda_2 - \lambda_1) t} = 0 \quad \text{while} \quad \lim_{t \to +\infty} p(t) = \pm \infty \quad \text{(as } c_k \text{ is pos. or neg.)}.
\]

So that we must have \(p(t) \equiv 0\) and then \(q(t) \equiv 0\). If \(\alpha > 0\) we repeat the same argument with the first limit replace by \(t \to -\infty\). Finally, in the case \(\alpha = 0\) we divide both sides of the equation by \(q(t)\) and collect real and imaginary parts to obtain

\[
r_1(t) + i r_2(t) = \frac{p(t)}{q(t)} = e^{\beta t} = \cos(\beta t) + i \sin(\beta t)
\]

where \(r_1(t)\) and \(r_1(t)\) are rational functions with real coefficients. Equating real and imaginary parts we see that this would imply that

\[
r_1(t) = \cos(\beta t), \quad r_2(t) = \sin(\beta t)
\]

which is impossible unless \(r_1(t) = 0\) and \(r_2(t) = 0\) since the right side has infinitely many zeros while the left can have only a finite number. This in turn implies that \(p(t) = 0\) and also \(q(t) = 0\). \(\square\)
Lemma 3.2.10. If $\ell > 0$, $\lambda_1 \neq \lambda_2$ are real or complex numbers and
\[
(D - \lambda_2)^\ell (p(t)e^{\lambda_1 t}) = 0
\]
where $p(t)$ is a polynomial, then $p(t) \equiv 0$.

Proof. We know that every solution of $(D - \lambda_2)^\ell y = 0$ can be written as $y = (q(t)e^{\lambda_2 t})$ for some polynomial $q(t)$ of degree at most $(\ell - 1)$. So the equation is whether or not there exists a polynomial $q(t)$ so that
\[
p(t)e^{\lambda_1 t} = q(t)e^{\lambda_2 t}.
\]
We note that this is only possible when $p(t) = 0$ and $q(t) = 0$ by Lemma 3.2.9.

Lemma 3.2.11. If $p(t)$ is any polynomial of degree less than or equal $(n - 1)$ then
\[
(D - \lambda_1)^m (p(t)e^{\lambda_2 t}) = q(t)e^{\lambda_2 t}
\]
where $q(t)$ is a polynomial of degree at most the degree of $p(t)$.

Proof. Consider the case $m = 1$. We have
\[
(D - \lambda_1) (p(t)e^{\lambda_2 t}) = q(t)e^{\lambda_2 t}
\]
where $q(t) = p'(t) + (\lambda_2 - \lambda_1)p(t)$ which is a polynomial of degree $p(t)$. You can now iterate this result for general $\ell > 0$.

Lemma 3.2.12. If $L(y) = y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y$ has real coefficients and $p(r) = r^n + a_1 r^{n-1} + \cdots + a_n$. Then $p(z) = p(\overline{z})$ for all $z \in \mathbb{C}$. Therefore if $p(\alpha + \beta i) = 0$ then $p(\alpha - \beta i) = 0$.

Proof. For every $z_1, z_2 \in \mathbb{C}$, we have $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$ and $\overline{z_1 z_2} = \overline{z_1} \overline{z_2}$ which also implies $\overline{z_1^n} = \overline{z_1}^n$.

From Lemma 3.2.12 we know that for a differential operator $L$ with real coefficients, all complex roots must occur in complex conjugate pairs (counting multiplicity) and from Lemma 3.2.8 we know that for a pair of complex roots $\lambda = \alpha + \beta i$ each of multiplicity $k$, a set of $2k$ linearly independent solutions is given for $j = 0, \cdots, (k - 1)$ by
\[
t^j e^{\lambda t} = t^j e^{\alpha t} \left( \cos(\beta t) + i \sin(\beta t) \right),
\]
\[
t^j e^{\lambda t} = t^j e^{\alpha t} \left( \cos(\beta t) - i \sin(\beta t) \right).
\]
From this we see that there is a set of real solutions given as a linear combination of these
solutions by
\[ t^j e^{\alpha t} \cos(\beta t) \] = \frac{1}{2} t^j \left( e^{\lambda t} + e^{\bar{\lambda} t} \right),
and
\[ x^j e^{\alpha t} \sin(\beta t) = \frac{1}{2\iota} t^j \left( e^{\lambda t} - e^{\bar{\lambda} t} \right). \]

We already know from Lemma 3.2.8 that \( t^j e^{\lambda t} \) and \( t^j e^{\bar{\lambda} t} \) are linearly independent. Suppose
we have a linear combination
\[ c^j t^j e^{\alpha t} \cos(\beta t) + d^j t^j e^{\alpha t} \sin(\beta t) = 0. \]
This would imply that
\[ \frac{(c^j - d^j i)}{2} t^j e^{\lambda t} + \frac{(c^j + d^j i)}{2} t^j e^{\bar{\lambda} t} = 0, \]
but since these functions are independent this implies
\[ (c^j - d^j i) = 0, \quad (c^j + d^j i) = 0, \quad \text{which implies} \quad c^j = d^j = 0. \]

Combining these results we have the main theorem:

**Theorem 3.2.13.** If \( L = y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y \) has real coefficients and we assume that
the polynomial \( p(r) = r^n + a_1 r^{(n-1)} + \cdots + a_n \) has zeros given by
\[ r_1, r_1^*, r_2, r_2^*, \cdots, r_f, r_f^*, r_{2f+1}, \cdots, r_s \]
where \( r_j = \alpha_j + \beta_j i, \quad j = 1, \cdots, f, \quad \alpha_j, \beta_j \in \mathbb{R}, \quad \beta_j \neq 0 \) and \( r_j \) for \( j = 2f+1, \cdots, s \) are real.
Let \( r_j \) have multiplicity \( m_j \) for all \( j \). Then if \( p_j(t) \) and \( q_j(t) \) denote arbitrary polynomials
(with real coefficients) of degree \( m_j - 1 \), the general solution of \( Ly = 0 \) can be written as
\[ y = \sum_{j=1}^{f} e^{\alpha_j t} \left[ p_j(t) \cos(\beta_j t) + q_j(t) \sin(\beta_j t) \right] + \sum_{j=2f+1}^{s} p_j(t) e^{r_j t}. \]

**Proof.** We need only prove that all the functions making up this general linear combination
are linearly independent. We already know that each particular term, i.e., a term of the form \( p_j(t) e^{r_j t} \) or \( e^{\alpha_j t} \left[ p_j(t) \cos(\beta_j t) + q_j(t) \sin(\beta_j t) \right] \) consists of linearly independent functions.
Note also that by rewriting this last expression in terms of complex exponentials, we have
the functions \( p_j(t) e^{r_j t} \) and \( p_j(t) e^{\bar{r}_j t} \). Thus let us suppose that we have a general linear
combination of the form
\[ \sum_{j=1}^{m} p_j(t) e^{r_j t} = 0, \quad \text{for some} \quad m, \]
where all we assume is that \( r_i \neq r_j \) for \( i \neq j \). We want to show this implies that every
polynomial \( p_j \equiv 0 \). We prove this by induction:
1. The case $s = 1$ have already done.

2. Assume that the statement holds for $s = k - 1$, i.e., $\sum_{j=1}^{k-1} p_j(t)e^{r_jt} = 0$ implies that every $p_j(t) \equiv 0$.

3. Assume that $\sum_{j=1}^{k} \left( p_j(t)e^{r_jt} \right) = 0$. We now apply $(D - r_k)^{m_k}$ to this expression and note that $(D - r_k)^{m_k} \left( p_k(t)e^{r_jt} \right) = 0$ so that the sum reduces to

$$\sum_{j=1}^{k-1} (D - r_k)^{m_k} \left( p_j(t)e^{r_jt} \right) = 0.$$ 

By Lemma 3.2.11 this sum can be written as

$$\sum_{j=1}^{k-1} q_j(t)e^{r_jt} = 0$$

where

$$(D - r_k)^{m_k} \left( p_j(t)e^{r_jt} \right) = q_j(t)e^{r_jt}$$

By the induction hypothesis we have $q_j(t) = 0$ for all $j = 1, \cdots, (k - 1)$. But this implies that

$$(D - r_k)^{m_k} \left( p_j(t)e^{r_jt} \right) = 0, \quad j = 1, \cdots, (k - 1)$$

which by Lemma 3.2.10 implies that $p_j(t) = 0$ for all $j = 1, \cdots, (k - 1)$. Finally we see that the original expression reduces to

$$\left( p_k(t)e^{r_kt} \right) = 0$$

which implies that $p_k(t) = 0$.

\[\square\]

**Method of Undetermined Coefficients**

As we have already learned, the method of variation of parameters provides a method of representing a particular solution to a nonhomogeneous linear problem

$$Ly = y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{(n-1)} y^{(1)} + a_n y = f$$
in terms of a basis of solutions \( \{y_j\}_{j=1}^n \) of the linear homogeneous problem. In the special case in which the operator \( L \) has constant coefficients, we have just seen that it is possible to construct such a basis of solutions for the homogeneous problem. Thus given any \( f \) we can write out a formula for a particular solution in integral form

\[
y_p(t) = \int_{t_0}^{t} g(t, s) f(s) \, ds.
\]

Unfortunately, the method of variation of parameters often requires much more work than is needed. As an example consider the problem

\[
Ly = y''' + y'' + y' + y = 1
\]
\[
y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 0.
\]

**Example 3.2.14.** For the homogeneous problem we have

\[
(D^3 + D^2 + D + 1)y = (D + 1)(D^2 + 1)y = 0
\]

so we can take

\[
y_1 = \cos t, \quad y_2 = \sin t, \quad y_3 = e^{-t}.
\]

Thus the wronskian is

\[
W(t) = \begin{vmatrix}
\cos t & \sin t & e^{-t} \\
-\sin t & \cos t & -e^{-t} \\
-cos t & -\sin t & e^{-t}
\end{vmatrix}
\]

and we can apply Abel’s theorem to obtain

\[
W(t) = W(0)e^{-\int_0^t 1 \, ds} = \begin{vmatrix}
1 & 0 & 1 \\
0 & 1 & -1 \\
-1 & 0 & 1
\end{vmatrix} e^{-t} = 2e^{-t}.
\]

Thus by the variation of parameters formula \( y_p = u_1 y_1 + u_2 y_2 \) where

\[
u_1' = \frac{1}{2} e^t \begin{vmatrix}
0 & \sin t & e^{-t} \\
0 & \cos t & -e^{-t} \\
1 & -\sin t & e^{-t}
\end{vmatrix} = -\frac{1}{2} (\cos t + \sin t),
\]
which implies
\[ u_1(t) = \frac{1}{2} (\cos(t) - \sin(t)). \]

Similarly, we obtain
\[ u_2'(t) = \frac{1}{2} (\cos t - \sin t), \quad u_3(t) = \frac{1}{2} e^t, \]
which imply
\[ u_2''(t) = \frac{1}{2} (\cos t + \sin t), \quad u_3(t) = \frac{1}{2} e^t, \]
So we get
\[ y_p = u_1y_1 + u_2y_2 \]
\[ = \frac{1}{2} (\cos t - \sin t) \cos t + \frac{1}{2} (\sin t + \cos t) \sin t + \frac{1}{2} e^t e^{-t} \]
\[ = 1 \]
Well yes, in retrospect we see that it would have been easy to see that \( y_p = 1 \) is a particular solution.

Now the general solution is
\[ y = 1 + c_1 \cos t + c_2 \sin t + c_3 e^{-t} \]
and we can apply the initial conditions to determine the constants which yields
\[ y = 1 + \frac{1}{2} (\sin t - \cos t - e^{-t}). \]

We note that if we were to apply the method for finding a particular solution with the properties
\[ y_p(0) = 0, \quad y_p'(0) = 0, \quad y_p''(0) = 0, \]
(as given in the proof of the n order case), then we would get
\[ y_p(t) = 1 - \frac{1}{2} (\cos t + \sin t + e^{-t}). \]
We note that the second term is part of the homogeneous solution so it can be excluded.

In any case this is a lot of work to find such a simple particular solution. In case the function \( f \) is a linear combination of:

1. polynomials,
2. polynomials times exponentials or,

3. polynomials times exponentials times sine or cosine,

i.e., if \( f \) is a solution of a linear constant coefficient homogeneous differential equation, one can apply the method of undetermined coefficients.

The method goes as follows:

1. Let \( L = P(D) = D^n + \sum_{j=1}^{n} a_j D^{n-j} \)

2. Let \( Ly = 0 \) have general solution \( y_h = \sum_{j=1}^{n} c_j y_j \).

3. Assume that \( M = Q(D) = D^m + \sum_{j=1}^{m} b_j D^{m-j} \) is a constant coefficient linear differential operator such that \( Mf = 0 \).

4. Then \( \tilde{L} = ML \) is a polynomial constant coefficient differential operator.

5. If \( y = y_h + y_p \) where \( y_p \) is a particular solution of \( Ly = f \) and \( y_h \) is the general solution of the homogeneous problem, then we have \( \tilde{L}y = 0 \).

6. On the other hand we can write the general solution of this problem by simply factoring \( \tilde{L} \) and applying the results of the previous section.

7. Note that the solution \( y_h = \sum_{j=1}^{n} c_j y_j \) is part of this general solution.

8. So let us denote the general solution by

\[
y = \sum_{j=1}^{n} c_j y_j + \sum_{j=1}^{m} d_j w_j.
\]

9. Now we also know, by the variation of parameters formula, that there exists a particular solution of \( Ly_p = f \).

10. This particular solution must also be a part of the full general solution of the large homogeneous problem, i.e., \( y_p = \sum_{j=1}^{n} c_j y_j + \sum_{j=1}^{m} d_j w_j \).
11. We know that $Ly_h = 0$ so we can choose $y_p = \sum_{j=1}^{m} d_j w_j$.

**Example 3.2.15.** Example: $Ly = (D^2 - 2D + 2)y = t^2 e^t \sin(t)$:

The general solution of the homogeneous equation $Ly = 0$ is

$$y_h = c_1 e^t \cos(t) + c_2 e^t \sin(t)$$

According to the above discussion we seek a differential operator $M$ so that

$$M(t^2 e^t \sin(t)) = 0.$$ 

We immediately choose

$$M = (D^2 - 2D + 2)^3$$

and we need to compute the general solution to the homogeneous problem

$$MLy = (D^2 - 2D + 2)^3(D^2 - 2D + 2)y = (D^2 - 2D + 2)^4y = 0,$$

which implies

$$y = (c_1 + c_2 t + c_3 t^2 + c_4 t^3)e^t \cos(t) + (d_1 + d_2 t + d_3 t^2 + d_4 t^3)e^t \sin(t).$$

If we now remove the part of this function corresponding to the solution of the homogeneous problem $Ly = 0$, we have

$$y_p = (at^3 + bt^2 + ct)e^t \cos(t) + (dt^3 + ft^2 + gt)e^t \sin(t)$$

After a lengthy calculation the first derivative

$$y' = ((d + a) t^3 + (3 a + b + f) t^2 + (2 b + c + g) t + c) e^t \cos(t) + ((-a + d) t^3 + (-b + f + 3 d) t^2 + (-c + g + 2 f) t + g) e^t \sin(t)$$

and the second derivative

$$y'' = (2 dt^3 + (6 d + 6 a + 2 f) t^2 + (6 a + 4 b + 4 f + 2 g) t + 2 b + 2 c + 2 g) e^t \cos(t) + (6 a + 4 b + 4 f + 2 g) t + 2 b + 2 c + 2 g) e^t \sin(t)$$

$$+ ( -2 a t^3 + (-6 a + 6 d - 2 b) t^2 + (-4 b + 4 f + 6 d - 2 c) t + 2 f + 2 g - 2 c) e^t \sin(t)$$
Plugging all of this into the equation yields

\[ y'' - 2y' + 2y = (6 dt^2 + (6 a + 4 f) t + 2 g + 2 b) e^t \cos(t) + (-6 at^2 + (-4 b + 6 d) t - 2 c + 2 f) e^t \sin(t) \]

Equating coefficients with the right hand side leads to the equations

\[
\begin{align*}
6 d &= 0 \\
6 a + 4 f &= 0 \\
2 g + 2 b &= 0 \\
-6 a &= 1 \\
-4 b + 6 d &= 0 \\
-2 c + 2 f &= 0
\end{align*}
\]

which have the solutions \( a = -1/6, \ b = 0, \ c = 1/4, \ d = 0, \ f = 1/4 \) and \( g = 0 \). Thus, a particular solution of the equation is

\[ y = \left( -\frac{t^3}{6} + \frac{t}{4} \right) e^t \cos(t) + \frac{t^2}{4} e^t \sin(t) \]

The following table contains a guide for generating a particular solution when one applies the method of undetermined coefficients. In particular, consider

\[ Ly = f. \]

<table>
<thead>
<tr>
<th>( P_m(t) = c_0 t^m + \cdots + c_m )</th>
<th>( x^s(a_0 t^m + \cdots + a_m) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P_m(t)e^{at} )</td>
<td>( t^s(a_0 t^m + \cdots + a_m)e^{at} )</td>
</tr>
</tbody>
</table>
| \( P_m(t)e^{at} \left\{ \begin{array}{l}
\sin \beta t \\
\cos \beta t
\end{array} \right\} \) | \( t^s e^{at} \left[ (a_0 t^m + \cdots + a_m) \cos \beta t + (b_0 t^m + \cdots + b_m) \sin \beta t \right] \) |

**Example 3.2.16.** Returning to Example 3.2.14 we see that the operator \( M = D \) annihilates \( f = 1 \) so we seek a particular solution \( y_p = 1 \).
3.3 Linear Systems

Recall that
\[ z^{(n)} + \alpha_1(t)z^{(n-1)} + \cdots + \alpha_n z = \beta(t) \]
maybe written as a 1st order nonhomogeneous system (hereafter referred to as \( \text{(LNH)} \))
\[ y'(t) = A(t)y(t) + B(t) \]

where
\[
A = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \vdots \\
\vdots & \cdots & \cdots & \cdots & \vdots \\
-\alpha_n & -\alpha_{n-1} & \cdots & -\alpha_1
\end{pmatrix}, \quad B = \begin{pmatrix}
0 \\
\vdots \\
0 \\
\beta(t)
\end{pmatrix}.
\]

Here we will study the more general case of \( \text{(LNH)} \) when \( A \) is a general \( n \times n \) matrix and \( B \) is a general \( n \times 1 \) vector function:
\[
A = \begin{pmatrix}
a_{11}(t) & \cdots & a_{1n}(t) \\
a_{21}(t) & \cdots & a_{2n}(t) \\
\vdots & \cdots & \vdots \\
a_{n1}(t) & \cdots & a_{nn}(t)
\end{pmatrix}, \quad B(t) = \begin{pmatrix}
b_1(t) \\
b_2(t) \\
\vdots \\
b_n(t)
\end{pmatrix}.
\]

with the assumption that the entries of \( A \) and \( B \) are all in \( C^0[a,b] \).

To study the \( \text{(LNH)} \) problem we follow the proceed just as in the \( n \)th order case by first studying the Linear Homogeneous \( \text{(LH)} \) problem, i.e., the case \( B = 0 \).
\[ y' = Ay. \quad (3.3.1) \]

**Definition 3.3.1.** A set of vectors \( \{y_j\}_{j=1}^n \subset \mathbb{R}^n \) is linearly independent on \([a,b]\) if
\[ \sum_{j=1}^n c_j y_j(t) = 0, \text{ for all } t \in [a,b], \text{ implies } c_1 = c_2 = \cdots = c_n = 0. \]

The vectors are said to linearly dependent on \([a,b]\) if there exists numbers \( \{c_j\}_{j=1}^n \) not all zero such that
\[ \sum_{j=1}^n c_j y_j(t) = 0, \text{ for all } t \in [a,b]. \]
Below we will use the following notation for the components of the vectors \( \{ y_j \}_{j=1}^n \):

\[
\begin{bmatrix}
y_{11} \\
y_{21} \\
y_{31} \\
\vdots \\
y_{n1}
\end{bmatrix}, \quad
\begin{bmatrix}
y_{12} \\
y_{22} \\
y_{32} \\
\vdots \\
y_{n2}
\end{bmatrix}, \quad \ldots, \quad
\begin{bmatrix}
y_{1j} \\
y_{2j} \\
y_{3j} \\
\vdots \\
y_{nj}
\end{bmatrix}, \quad \ldots, \quad
\begin{bmatrix}
y_{1n} \\
y_{2n} \\
y_{3n} \\
\vdots \\
y_{nn}
\end{bmatrix}
\]

A linearly independent set \( \{ y_j \}_{j=1}^n \subset \mathbb{R}^n \) of solutions to the \((LH)\) problem (3.3.1) is called a fundamental set. Given a fundamental set we define a fundamental matrix by

\[
\Phi(t) = \begin{bmatrix}
y_{11}(t) & \cdots & y_{1n}(t) \\
y_{21}(t) & \cdots & y_{2n}(t) \\
\vdots & \vdots & \vdots \\
y_{n1}(t) & \cdots & y_{nn}(t)
\end{bmatrix}
\tag{3.3.2}
\]

**Theorem 3.3.2.** [Abel’s Formula] If \( y_1(t), \ldots, y_n(t) \) are solutions of \((LH)\) and \( t_0 \in (a, b) \), then

\[
W(t) = W(t_0) \exp \left( \int_{t_0}^{t} \text{tr} A(s) ds \right), \quad \text{where} \quad \text{tr}(A)(s) = \sum_{j=1}^{n} a_{jj}(s).
\]

Thus the determinant of \( \{ y_1, \ldots, y_n \} \) is never 0 or identically 0.

**Proof.** We give the proof for \( n = 2 \). The method of proof is identical for higher dimensions. Note that

\[
W'(t) = \begin{vmatrix}
y_{11}' & y_{12}' \\
y_{21}' & y_{22}'
\end{vmatrix} + \begin{vmatrix}
y_{11} & y_{12} \\
y_{21} & y_{22}
\end{vmatrix}
\]

\[
= \begin{vmatrix}
a_{11} \cdot y_{11} + a_{12} \cdot y_{21} & a_{11} \cdot y_{12} + a_{12} \cdot y_{22} \\
y_{21} & y_{22}
\end{vmatrix}
\]

\[
+ \begin{vmatrix}
y_{11} & y_{12} \\
a_{21} \cdot y_{11} + a_{22} \cdot y_{21} & a_{21} \cdot y_{12} + a_{22} \cdot y_{22}
\end{vmatrix}
\]

\[
= \begin{vmatrix}
a_{11} \cdot y_{11} & a_{11} \cdot y_{12} \\
y_{21} & y_{22}
\end{vmatrix} + \begin{vmatrix}
y_{11} & y_{12} \\
a_{22} \cdot y_{21} + a_{22} \cdot y_{22}
\end{vmatrix}
= a_{11}W(t) + a_{22}W(t) = \text{tr} A W(t).
\]
3.3. LINEAR SYSTEMS

Hence

\[ W(t) = W(t_0) \exp \left( \int_{t_0}^{t} \text{tr} A(s) \, ds \right). \]

The proof of the next theorem is identical to that of Theorem 3.2.3.

**Theorem 3.3.3.** Suppose \( \{y_1, \ldots, y_n\} \) are solutions of \((LH)\). If the functions are linearly independent on \([a, b]\), then \( W(t) \neq 0 \) for all \( t \in (a, b) \). Conversely, if there exists \( t_0 \) such that \( W(t_0) = 0 \), then \( W(t) \equiv 0 \) and the \( y_i(t) \) are linearly dependent on \((a, b)\).

**Theorem 3.3.4.**

1. A fundamental set exists.

2. The set of solutions \( S = \{y \in \mathbb{R}^n : y' = Ay\} \) to \((LH)\) forms an \( n \)-dimensional vector space.

This implies that if \( \{y_1, \ldots, y_n\} \) is a linearly independent set of solutions of \((LH)\), then given any solution \( y(t) \) of \((LH)\), there exists unique constants \( c_1, \ldots, c_n \) such that

\[ y(t) = c_1 y_1(t) + \cdots + c_n y_n(t). \]

**Proof.** For part 1) we need only let \( y_j \) denote the solution of

\[ y' = Ay \]

\[ y(t_0) = e_j \quad \text{\( j \)th standard unit basis vector in } \mathbb{R}^n. \]

Then \( W(t_0) = 1 \) and so the set \( \{y_1, \ldots, y_n\} \) is linearly independent.

For part 2), we show that the set \( \{y_1, \ldots, y_n\} \) from part 1) forms a basis for \( S \). We have already shown that it is a linearly independent set so we need only show that it is a spanning set. To this end, let \( z(t) \) be a solution of \((LH)\) and let \( z(t_0) = z^0 = [z_1^0 \ldots z_n^0]^T \in \mathbb{R}^n \). Let \( \Phi(t) \) be the fundamental matrix for the fundamental set \( \{y_1, \cdots, y_n\} \) given above. Then note that

\[ z^0 = \begin{bmatrix} z_1^0 \\ z_2^0 \\ \vdots \\ z_n^0 \end{bmatrix} = z_1^0 e_1 + z_2^0 e_2 + \cdots + z_n^0 e_n. \]

Consider the vector

\[ y(t) \equiv z_1^0 y_1(t) + z_2^0 y_2(t) + \cdots + z_n^0 y_n(t). \]
Then, as a sum of solutions of (LH) we have that $y$ is also a solution of (LH) and at $t = t_0$ we have
\[ y(t_0) = z_1^0 y_1(t_0) + z_2^0 y_2(t_0) + \cdots + z_n^0 y_n(t_0) = z_1^0 e_1 + z_2^0 e_2 + \cdots + z_n^0 e_n = z^0 = z(t_0). \]

Now by the fundamental existence uniqueness theorem $z(t) = y(t)$ for all $t$. Thus the set is a spanning set and we see that $S$ is an $n$-dimensional vector space.

Note that, in the proof of the last theorem the set of linearly independent solutions could have been any such set. In particular, if $\{y_j\}_{j=1}^n$ is any linearly independent set of solutions to (LH) with fundamental matrix $\Phi(t)$ (which is therefore nonsingular), then given any solution $z(t)$ of (LH) with $z(t_0) = z^0$, there exists $c_1, \cdots, c_n$ so that
\[ z(t) = c_1 y_1 + \cdots + c_n y_n = \Phi(t) \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \equiv \Phi(t)C. \]

Namely,
\[ C = \Phi^{-1}(t_0)z(t_0) = \Phi^{-1}(t_0)z^0. \]

Thus
\[ z(t) = \Phi(t)\Phi^{-1}(t_0)z^0. \]

Given any fundamental matrix $\Phi(t)$, let
\[ \Psi(t) = \Phi(t)\Phi^{-1}(t_0) \]
then any solution $z(t)$ of (LH) can be written as
\[ z(t) = \Psi(t)z^0 \]
and we have
\[ \det \Psi(t_0) = \det(\Psi(t_0)\Psi(t_0)^{-1}) = \det(I) = 1. \]

For the special fundamental set $\{y_1, \cdots, y_n\}$ satisfying
\[ y_j(t_0) = e_j, \]
we have,
\[ \Psi(t) = \Phi(t) \]
since $\Phi(t_0) = I$. 

Suppose \( \{y_1, \cdots, y_n\} \) are linearly independent solutions of (LH). Let
\[
\Psi(t) = \begin{bmatrix} y_1 & \cdots & y_n \end{bmatrix}.
\]
Then
\[
\Psi'(t) = \begin{bmatrix} y'_1 & \cdots & y'_n \end{bmatrix}
= \begin{bmatrix} Ay_1 & \cdots & Ay_n \end{bmatrix}
= A \begin{bmatrix} y_1 & \cdots & y_n \end{bmatrix}
= A\Psi(t).
\]

We call \( \Psi \) a solution matrix. The next theorem answers the question: when is a solution matrix a fundamental matrix?

**Theorem 3.3.5.** A solution matrix \( \Psi \) of (LH) is a fundamental matrix iff \( \det \Phi(t) \neq 0 \) for all \( t \in (a, b) \).

**Proof.** Since the columns of \( \Psi \) are solutions of (LH), we know \( \{y_1, \cdots, y_n\} \) is a fundamental set iff
\[
\det \Psi(t) \neq 0 \text{ for all } t
\]

\( \square \)

**Theorem 3.3.6.** If \( \Phi(t) \) is a fundamental matrix and \( C \) is a nonsingular constant matrix, then \( \Phi(t)C \) is a fundamental matrix

**Proof.** Let \( C = \begin{bmatrix} C^1 & C^2 & \cdots & C^n \end{bmatrix} \) denote a column delimited matrix and compute
\[
(\Phi(t)C)' = \left( \begin{bmatrix} \Phi(t)C^1 & \Phi(t)C^2 & \cdots & \Phi(t)C^n \end{bmatrix} \right)'
= \begin{bmatrix} A\Phi(t)C^1 & A\Phi(t)C^2 & \cdots & A\Phi(t)C^n \end{bmatrix}
= A\Phi(t)C.
\]
Moreover,
\[
\det(\Phi C) = \det(\Phi(t))(\det C) \neq 0.
\]

\( \square \)

**Theorem 3.3.7.** If \( \Phi \) and \( \Psi \) are fundamental matrices for (LH), then there is a nonsingular matrix \( C \) so that
\[
\Psi = \Phi C
\]
CHAPTER 3. LINEAR EQUATIONS

**Proof.** Let

\[ \Psi = [\psi_1 \cdots \psi_n], \quad \Phi = [\varphi_1, \cdots, \varphi_n]. \]

Since the sets of vectors \( \{\psi_j\}_{j=1}^n \) and \( \{\varphi_j\}_{j=1}^n \) are bases for \( \mathbb{R}^n \), given the column vectors \( \psi_j \) there exists constant vectors \( c_{1j}, \ldots, c_{nj} \) such that

\[
\psi_j = c_{1j}\varphi_1 + \cdots + c_{nj}\varphi_n
\]

\[
= (\varphi_1, \cdots, \varphi_n) \begin{pmatrix} c_{1j} \\ \vdots \\ c_{nj} \end{pmatrix}
\]

\[
= \Phi C_j
\]

Take

\[
C = [C_1 \cdots C_n].
\]

Then

\[
\Psi = \Phi C.
\]

Since \( \det \Phi \det C = \det \Psi \neq 0 \), the result follows.

\[ \square \]

**Nonhomogeneous Linear Systems**

We now consider the nonhomogeneous linear system (LNH)

\[
y' = Ay + B.
\]

To construct a solution, we imitate the method of variation of parameters. That is, we seek a particular solution

\[
y_p(t) = \Phi(t)v(t), \quad v(t) \in \mathbb{R}^n.
\]

Then

\[
y_p'(t) = \Phi'v + \Phi v' = A\Phi v + \Phi v' = Ay_p + \Phi v'.
\]

Thus we need

\[
\Phi v' = B \quad \Rightarrow \quad v' = \Phi^{-1}B
\]

or

\[
v(t) - v(t_0) = \int_{t_0}^{t} \Phi^{-1}(s)B(s)ds,
\]

and we can take the solution that is zero at \( t_0 \) to be

\[
v(t) = \int_{t_0}^{t} \Phi^{-1}(s)B(s)ds.
\]
Thus
\[ y_p = \Phi(t) \int_{t_0}^{t} \Phi^{-1}(s)B(s)ds. \]

Now, if \( y(t) \) is any other solution of (LNH), then \( y(t) - y_p(t) \) satisfies (LH) and hence
\[ y(t) - y_p(t) = \Phi(t)C \]
for some constant vector \( C \). Therefore the general solution of LNH is
\[ y(t) = \Phi(t)C + \Phi(t) \int_{t_0}^{t} \Phi^{-1}(s)B(s)ds \]
which we rewrite as
\[ y(t) = \Phi(t)C + \Phi(t) \int_{t_0}^{t} \Phi^{-1}(s)ds. \]

For the IVP
\[ y(t_0) = y_0, \]
then
\[ y(t_0) = \Phi(t_0)C \Rightarrow C = \Phi^{-1}(t_0)y_0. \]

We have derived the **Variation of Parameters Formula**
\[ y(t) = \Phi(t)\Phi^{-1}(t)y_0 + \int_{t_0}^{t} \Phi(t)\Phi^{-1}(s)B(s)ds. \]

If \( A \) is constant, it is left as an exercise to show that the formula becomes
\[ y(t) = \Phi(t-t_0)y_0 + \int_{t_0}^{t} \Phi(t-s)B(s)ds, \]

where \( \Phi(0) = I \).

### 3.4 Linear Systems with Constant Coefficients

We conclude this chapter by examining the special case that the coefficient matrix \( A \) is constant. As in the scalar case, we work over the complex numbers and specialize to the real case.

Here is one idea to solve the system
\[ y' = Ay. \]
Seek solutions of the form $y(t) = e^{\lambda t}v$ where $v$ a constant vector. Then

$$y' = \lambda e^{\lambda t}v$$

and

$$Ay = A(e^{\lambda t}v) = e^{\lambda t}Av.$$  

Thus we get a solution if

$$\lambda e^{\lambda t}v = e^{\lambda t}Av$$

or

$$Av = \lambda v,$$

that is $(\lambda, v)$ must be an eigenpair. When we constructed solutions to the linear scalar differential equation we considered cases as determined by the roots of the characteristic polynomial. For systems, cases are distinguished by the roots of the characteristic polynomial of $A$.

It’s not hard to do the analysis in the case where the matrix can be diagonalized.

**Theorem 3.4.1.** Suppose $A$ is $n \times n$ and has distinct $\epsilon$-values $\{\lambda_1, \ldots, \lambda_n\}$, then there exist $n$ linearly independent $\epsilon$-vectors $\{v_1, \ldots, v_n\}$. In this case,

$$\{e^{\lambda_1 t}v_1, \ldots, e^{\lambda_n t}v_n\}$$

is a fundamental set.

**Proof.** It is clear, since each $\lambda_j$ is a simple eigenvalue, that for each $j$ there must be a nonzero solution to the equation $(\lambda_j - A)v = 0$ since $\det(\lambda_j - A) = 0$. For each $j$ let this solution be called $v_j$. Now let us suppose that

$$c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0.$$  

On multiplying this equation by $A$ and using $Av_j = \lambda_j v_j$ we have

$$c_1\lambda_1 v_1 + c_2\lambda_2 v_2 + \cdots + c_n\lambda_n v_n = 0.$$  

Repeating we arrive at the system

$$c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0$$

$$c_1\lambda_1 v_1 + c_2\lambda_2 v_2 + \cdots + c_n\lambda_n v_n = 0$$

$$\vdots$$

$$c_1\lambda_1^{n-1} v_1 + c_2\lambda_2^{n-1} v_2 + \cdots + c_n\lambda_n^{n-1} v_n = 0.$$
Defining the Vandermonde matrix
\[ V = \begin{bmatrix}
1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{(n-1)} \\
1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{(n-1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \lambda_n & \lambda_n^2 & \cdots & \lambda_n^{(n-1)}
\end{bmatrix} \]
we note that the determinant of this Vandermonde is zero if and only if \( \lambda_i = \lambda_j \) for some \( i \) and \( j \). Under the assumption that the eigenvalues are distinct, we see that \( V \) is nonsingular. So the system above can be written, first as
\[ \begin{bmatrix} c_1 v_1 & c_2 v_2 & \cdots & c_n v_n \end{bmatrix} V = 0, \]
and then, on multiplying by \( V^{-1} \) on the right,
\[ \begin{bmatrix} c_1 v_1 & c_2 v_2 & \cdots & c_n v_n \end{bmatrix} = 0, \]
which implies that
\[ c_j v_j = 0, \quad j = 1, \ldots, n \quad \Rightarrow \quad c_j = 0, \quad j = 1, \ldots, n \]
since, by assumption, \( v_j \neq 0 \) for all \( j \).

To see that these vectors are linearly independent we note that
\[
W\{e^{\lambda_1 t} v_1, \ldots, e^{\lambda_n t} v_n\} = |e^{\lambda_1 t} v_1, \ldots, e^{\lambda_n t} v_n| \\
= e^{\lambda_1 t}|v_1, e^{\lambda_2 t} v_2, \ldots, e^{\lambda_n t} v_n| \\
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ Quad
Proof. \((\Rightarrow)\) Assume that there exists \(P\) such that \(P^{-1} = P^*\) and \(A = PDP^*\) with \(D\) a diagonal matrix. Then \(A^* = PD^*P^*\) and we have
\[
AA^* = (PDP^*)(PD^*P^*) = P(DD^*)P^* = P(D^*D)P^* = (PD^*P^*)(PDP^*) = A^*A
\]
where we have used the fact that diagonal matrices commute. Thus \(A\) is normal.

\((\Leftarrow)\) For this direction we need a well known result in matrix theory – Schur’s Theorem.

**Theorem** For every \(n \times n\) matrix \(A\) there exists a unitary matrix \(P\) such that \(A = PTP^*\) with \(T\) an upper triangular matrix.

The proof of this theorem is very simple but will not be included here. A proof of this theorem can be found in any book on matrix theory (see, for example, *Matrix Theory* by James M. Ortega, published by Plenum Publishing Co.). Assuming Schur’s theorem we have that there exists an upper triangular matrix \(T\) so that \(T = P^*AP\) and, assuming that \(A\) is normal, we show that \(T\) is also normal. We have
\[
TT^* = (P^*AP)(P^*A^*P) = P^*(AA^*)P = T^*T,
\]
and we see that \(T\) is normal. But a normal upper-triangular matrix must be a diagonal matrix. We consider the \(2 \times 2\) case:
\[
\begin{bmatrix}
 a_{11} & a_{12} \\
 0 & a_{21}
\end{bmatrix}
\begin{bmatrix}
 a_{11} & 0 \\
 a_{12} & a_{21}
\end{bmatrix}
= \begin{bmatrix}
 a_{11} & 0 \\
 a_{12} & a_{21}
\end{bmatrix}
\begin{bmatrix}
 a_{11} & a_{12} \\
 0 & a_{21}
\end{bmatrix}.
\]
From the (1,1)-entry we have
\[
|a_{11}|^2 + |a_{12}|^2 = |a_{11}|^2
\]
which implies that \(a_{12} = 0\) so \(T\) is diagonal.

Now in case \(A = PDP^*\) then \(AP = PD\). Let us designate \(P\) as a column delimited matrix by \(P = [P_1 \ P_2 \ \cdots \ P_n]\). The we can write \(AP = PD\) as
\[
\begin{bmatrix}
 AP_1 & AP_2 & \cdots & AP_n
\end{bmatrix} = [\lambda_1 P_1 \ \lambda_2 P_2 \ \cdots \ \lambda_n P_n]
which implies $AP_j = \lambda_j P_j$ for all $j = 1, \ldots, n$. Also $A^*P = PD^*$ implies that $A^*P_j = \overline{\lambda_j}P_j$ for all $j = 1, \ldots, n$. So the eigenvalues of $A$ are $\{\lambda_j\}$ with eigenvectors $\{P_j\}$. To see that the eigenvectors are orthogonal just note that $P$ unitary implies $I = PP^* = P^*P$ which simply says that $<P_j, P_k> = \delta_{jk}$, i.e., the vectors $\{P_j\}$ are orthogonal.

Example 3.4.3.

$$y' = \begin{bmatrix} 1 & 12 \\ 3 & 1 \end{bmatrix} y$$

Here the e-pairs are

$$\left(7, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right), \left(-5, \begin{bmatrix} -2 \\ 1 \end{bmatrix}\right).$$

Two linearly independent solutions are

$$y_1(t) = e^{7t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad y_2(t) = e^{-5t} \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

Hence a fundamental matrix is

$$\Phi(t) = \begin{bmatrix} 2e^{7t} & -2e^{-5t} \\ e^{7t} & e^{-5t} \end{bmatrix}.$$  

Consider next the case of a real matrix with distinct eigenvalues, some of which are complex. We wish to find real solutions.

Suppose $\lambda = \alpha + i\beta$ is an e-value with an e-vector $v = v_1 + iv_2$. Then

$$e^{\lambda t}v$$

is a complex solution of (LH). We obtain two real solutions by writing

$$y(t) = e^{\alpha t}(\cos(\beta t) + i \sin(\beta t))(v_1 + iv_2)$$

$$= e^{\alpha t}(\cos(\beta t)v_1 - \sin(\beta t)v_2)$$

$$+ ie^{\alpha t}(\sin(\beta t)v_1 + \cos(\beta t)v_2)$$

and taking

$$y_1(t) = e^{\alpha t}(\cos(\beta t)v_1 - \sin(\beta t)v_2)$$

$$y_2(t) = e^{\alpha t}(\sin(\beta t)v_1 + \cos(\beta t)v_2).$$
Example 3.4.4. The characteristic equation for the system
\[ y' = Ay = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} y \]
is
\[ \varphi(t) = |A - \lambda I| = (1 - \lambda)((1 - \lambda)^2 + 1) = 0 \]
The e-values are \( \lambda_1 = 1, \lambda = 1 \pm i \). For \( \lambda_1 = 1 \) we see
\[ (A - I)v = 0 \implies v = c \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad c \in \mathbb{C}. \]
if \( \lambda = 1 + i \), then
\[ (A - (1 + i))v = \begin{bmatrix} -i & 0 & 0 \\ 0 & -i & -1 \\ 0 & 1 & -i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0 \]
and so we may take
\[ v_1 = 0, \quad iv_2 = -v_3 \text{ or } v_2 = i, \quad v_3 = 1. \]
or
\[ v = \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix}. \]
We get a solution
\[ y(t) = e^t(\cos(t) + i \sin(t)) \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix}. \]
Two linearly independent real solutions are
\[ y_2(t) = \Re(y(t)) = e^t \begin{bmatrix} 0 \\ -\sin(t) \\ \cos(t) \end{bmatrix}. \]
\[ y_3(t) = \Im(y(t)) = e^t \begin{bmatrix} 0 \\ \cos(t) \\ \sin(t) \end{bmatrix}. \]
A fundamental matrix is given by
\[
\Phi(t) = e^t \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos(t) & -\sin(t) \\
0 & \sin(t) & \cos(t)
\end{bmatrix}.
\]

These techniques leave open the question of what to do in the case of repeated roots. For this reason, and for theoretical reasons, we introduce the concept of the exponential of a matrix.

Recall that for a real number \( t \),
\[
e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} = 1 + t + \frac{t^2}{2} + \frac{t^3}{3!} + \cdots,
\]
and this series converge absolutely for all values of \( t \).

If \( A \) is an \( n \times n \) real or complex matrix, we define
\[
e^{At} = \sum_{j=0}^{\infty} \frac{1}{j!} t^j A^j = I + At + \frac{A^2 t^2}{2!} + \frac{A^3 t^3}{3!} + \cdots.
\]

To see that \( e^{At} \) is well defined, let
\[
S_k(t) = I + At + \cdots + \frac{A^k t^4}{k!}.
\]
then for any \( T > 0 \) and \( t \in [-T,T] \)
\[
\|S_m(t) - S_p(t)\| = \| \sum_{k=p+1}^{m} \frac{A^k t^k}{k!} \| \\
\leq \sum_{k=p+1}^{m} \frac{\|A\|^k T^k}{k!},
\]
where we may assume \( p < m \) without loss of generality. This can be made arbitrarily small by choosing \( p \) large. Hence the series converges. If we view the set of \( n \times n \) matrices as elements of \( K^{n \times n} \) the result follows since \( K^{n \times n} \) is complete.

**Theorem 3.4.5.** The matrix exponential satisfies the following properties.

1. For a zero matrix 0, \( e^0 = I \).
2. If $B$ commutes with $A$, then $B$ commutes with $e^{At}$. In particular, $A$ commutes with $e^{At}$. If $B$ commutes with $A$, then $e^B$ commutes with $e^{At}$.

3. The matrix valued function $S(t) = e^{At}$ is differentiable, with

$$\frac{d}{dt}e^{At} = Ae^{At} = e^{At}A.$$

4. If $A$ and $B$ commute, then $e^{At+Bt} = e^{At}e^{Bt} = e^{Bt}e^{At}$.

5. For real numbers $t$ and $s$, $e^{A(t+s)} = e^{At}e^{As}$.

6. For any matrix $A$, $e^{At}$ is invertible and $(e^{At})^{-1} = e^{-At}$.

7. If $P$ is invertible, then $e^{P^{-1}AtP} = P^{-1}e^{At}P$.

Proof. We will slightly alter the order of proof:

1) The first statement is obvious on setting $t = 0$ in the infinite series and noting that $A^0 = I$ and $0! = 1$.

2) By induction we can readily see that $BA = AB$ implies $BA^j = A^jB$ for all $j = 1, 2, \ldots$ which implies $S_k(t)B = BS_k(t)$ so given $\epsilon > 0$ we can choose $K > 0$ so that for $k > K$ we have $\|S_k(t) - S(t)\| \leq \epsilon/(2\|B\|)$ for all $t \in [-T, T]$ and we can write

$$\|BS(t) - S(t)B\| \leq \|BS(t) - S_k(t)B\| + \|S_k(t)B - S(t)B\|$$

$$= \|B(S(t) - S_k(t))\| + \|(S_k(t) - S(t))B\|$$

$$\leq 2\|B\||\|(S_k(t) - S(t))\| \leq \epsilon.$$

Since $\epsilon$ is arbitrary we are done.

3) We first recall a result from “baby reals”: If $\{u_j\} \subset C^1[a, b]$ and

$$\sum_{j=1}^{\infty} u_j(t) \quad \text{and} \quad \sum_{j=1}^{\infty} u'_j(t) \quad \text{converge uniformly on} \quad [a, b].$$

Then

$$\left(\sum_{j=1}^{\infty} u_j(t)\right)' = \sum_{j=1}^{\infty} u'_j(t).$$
3.4. LINEAR SYSTEMS WITH CONSTANT COEFFICIENTS

Now we need only note that

\[ S_k'(t) = \sum_{j=0}^{k} \frac{j! A^j t^{j-1}}{j!} = A \sum_{j=1}^{k} \frac{A^{j-1} t^{j-1}}{(j-1)!} = A \sum_{j=0}^{k-1} \frac{A^j t^j}{j!} = A S_{k-1}(t). \]

Since \( A \) is continuous and \( S_{k-1}(t) \) converges uniformly, we see that the \( S_k'(t) \) converges uniformly and we can apply the above result to conclude

\[ S'(t) = \lim_{k \to \infty} S_k'(t) = \lim_{k \to \infty} A S_k(t) = A \lim_{k \to \infty} S_k(t) = AS(t) = S(t)A. \]

5) To show that \( S(t) \) is invertible and to compute the inverse we use 4) and 1). Namely, \( I = S(0) = S(t-t) = S(t)S(-t) \) which implies that \( S(t) \) is invertible and its inverse satisfies \( (S(t))^{-1} = S(-t) \).

4) To prove that \( S(t_1 + t_2) = S(t_1)S(t_2) \) we let \( t_2 = s \) be fixed and consider \( t = t_1 \) as a variable. Then we define \( X(t) = e^{As} e^{At} \) and \( Y(t) = e^{A(t+s)} \). Then by part 3) we have

\[
\begin{align*}
X'(t) &= AX(t), \quad X(0) = e^{As} \\
Y'(t) &= AY(t), \quad Y(0) = e^{As}.
\end{align*}
\]

So by the fundamental existence and uniqueness theorem we have \( X(t) = Y(t) \) for all \( t \).

7) \( \Rightarrow \) Let \( Y(t) = e^{(A+B)t} \) and \( Z(t) = e^{At} e^{Bt} \). Note that \( Y(0) = Z(0) = I \) show we need only show that \( Y \) and \( Z \) satisfy the same linear ordinary differential equation.

First we note that

\[ Y'(t) = (A + B) Y(t) \]

is obvious and then by the product rule we have

\[ Z'(t) = Ae^{At} e^{Bt} + e^{At} Be^{Bt}. \]

But it was proved in 2) that \( AB = BA \) implies \( e^{At} B = Be^{At} \) so we have

\[ Z'(t) = (A + B) Z(t) \]

and we are done.

\( \Leftarrow \) Assume that \( Y(t) = Z(t) \) for all \( t \). Then \( Y'(t) = Z'(t) \) which gives

\[
(A + B) e^{At} e^{Bt} = (A + B) e^{(A+B)t} = Y'(t) = Z'(t) = Ae^{At} e^{Bt} + e^{At} Be^{Bt}
\]
This in turn implies
\[ Be^{At}e^{Bt} = e^{At}Be^{Bt} \]
Now apply \( e^{-Bt} \) on the left and use 5) to get
\[ Be^{At} = e^{At}B \]
which we differentiate with respect to \( t \) and then set \( t = 0 \) to obtain
\[ BAe^{At}\bigg|_{t=0} = Ae^{At}B \bigg|_{t=0} \Rightarrow BA = AB. \]

6) If we let \( B = P^{-1}AP \) want to show that \( \exp(Bt) = P^{-1} \exp(At)P \). Note is easy to see that \( B^j = P^{-1}A^jP \) and so
\[ \sum_{j=1}^{k} \frac{B^j t^j}{j!} = \sum_{j=1}^{k} \frac{P^{-1}A^jPt^j}{j!} = P^{-1}\left(\sum_{j=1}^{k} \frac{A^j t^j}{j!}\right)P, \]
and passing to the limit on each side we arrive at
\[ e^{Bt} = \sum_{j=1}^{\infty} \frac{B^j t^j}{j!} = P^{-1}\left(\sum_{j=1}^{\infty} \frac{A^j t^j}{j!}\right)P = P^{-1} \exp(At)P. \]

\[ \square \]

**Theorem 3.4.6.** \( \Phi(t) = e^{At} \) is a fundamental matrix for \((LH)\).

**Proof.** Since
\[ \Phi' = Ae^{At} = A\Phi, \]
it is a solution matrix. Since
\[ \det(\Phi(0)) = 1, \]
by Abel’s formula \( \det(\Phi(t)) \neq 0 \) for all \( t \) and hence by Theorem 3.3.5 it is fundamental. \( \square \)

Hence every solution of \((LH)\) when \( A \) is constant is given by
\[ y(t) = e^{At}c \]
for a suitably chosen constant vector \( c \). In fact, plugging in \( t = 0 \) we see that \( c = y(0) \).

Recall the nonhomogeneous problem is solved by the variation of parameters formula
\[ y' = Ay + B, y(0) = y_0 \Rightarrow y(t) = \Phi(t)\Phi^{-1}(t_0)y_0 + \int_{t_0}^{t} \Phi(t)\Phi^{-1}(s)B(s)ds. \]
In the case where $A$ is constant the variations of parameters formula becomes

$$y(t) = e^{At} e^{-A t_0} y_0 + \int_{t_0}^{t} e^{At} e^{-A s} b(s) \, ds$$

or

$$y(t) = e^{A(t-t_0)} y_0 + \int_{t_0}^{t} e^{A(t-s)} b(s) \, ds.$$

We now turn to the computation of $e^{At}$.

**Definition 3.4.7.** An $n \times n$ matrix $A$ is diagonalizable if and only if there exists a nonsingular matrix $P$ such that

$$P^{-1}AP = D = \text{diag} \left( \lambda_1, \lambda_2, \ldots, \lambda_n \right).$$

Here the $\{\lambda_j\}$ are the eigenvalues of $A$.

Note that if $A$ is diagonalizable then $AP = PD$ and this can be written, in terms of columns, as

$$[AP_1 \ AP_2 \ \cdots \ AP_n] = [\lambda_1 P_1 \ \lambda_2 P_2 \ \cdots \ \lambda_n P_n]$$

which implies that not only are the $\{\lambda_j\}$ the eigenvalues of $A$ but also the columns $P_j$ of $P$ are the associated eigenvectors. Notice also that since $P$ is nonsingular we know that the columns are linear independent which implies that the eigenfunctions are linearly independent. Also by part 6) of Theorem 3.4.5 we have

$$e^{At} = e^{P D t P^{-1}} = P e^{D t} P^{-1}.$$

Thus to compute $e^{At}$ we first consider the case $e^{Dt}$ with $D$ diagonal.

Suppose $D$ is diagonal, say

$$D = \begin{bmatrix} \lambda_1 & 0 \\ \vdots & \ddots \\ 0 & \lambda_n \end{bmatrix}$$

Then

$$e^{Dt} = I + \begin{bmatrix} \lambda_1 t & 0 \\ \vdots & \ddots \\ 0 & \lambda_n t \end{bmatrix} + \begin{bmatrix} \lambda_1^2 t^2 / 2! & 0 \\ \vdots & \ddots \\ 0 & \lambda_n^2 t^2 / 2! \end{bmatrix} + \cdots$$

$$= \begin{bmatrix} 1 + \lambda_1 t + \lambda_1^2 t^2 / 2! + \cdots & 0 \\ \vdots & \ddots \\ 0 & 1 + \lambda_n t + \lambda_n^2 t^2 / 2! + \cdots \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ \vdots & \ddots \\ 0 & e^{\lambda_n t} \end{bmatrix}.$$
So if \( A \) is diagonalizable, then

\[
P^{-1}AP = D = \begin{bmatrix}
\lambda_1 & 0 \\
& \ddots \\
0 & \lambda_n
\end{bmatrix},
\]

and so

\[
e^{At} = P \begin{bmatrix}
e^{\lambda_1 t} & 0 \\
& \ddots \\
0 & e^{\lambda_n t}
\end{bmatrix} P^{-1}
\]

Hence in this case

\[
e^{At} = P \begin{bmatrix}
e^{\lambda_1 t} & 0 \\
& \ddots \\
0 & e^{\lambda_n t}
\end{bmatrix} P^{-1} = [e^{\lambda_1 t} P_1 \ldots e^{\lambda_n t} P_n] P^{-1} = \Phi(t) P^{-1}
\]

where \( \Phi(t) \) is the matrix constructed in (3.4.3).

In the special case of normal matrix (e.g., a selfadjoint or symmetric matrix) for which the eigenvectors are orthogonal so that \( P^{-1} = P^T \) we see that

\[
e^{At} y_0 = \begin{bmatrix}
e^{\lambda_1 t} P_1 & \ldots & e^{\lambda_n t} P_n
\end{bmatrix} P^{-1} y_0 = \begin{bmatrix}
e^{\lambda_1 t} P_1 & \ldots & e^{\lambda_n t} P_n
\end{bmatrix} y_0
\]

\[
= \begin{bmatrix}
| P_1^T y_0 \\
P_2^T y_0 \\
\vdots \\
P_n^T y_0
\end{bmatrix} = \begin{bmatrix}
e^{\lambda_1 t} P_1 & \ldots & e^{\lambda_n t} P_n
\end{bmatrix} \begin{bmatrix}
| \langle y_0, P_1 \rangle \\
\langle y_0, P_2 \rangle \\
\vdots \\
\langle y_0, P_n \rangle
\end{bmatrix}
\]

\[
= \sum_{j=1}^{n} e^{\lambda_j t} \langle y_0, P_j \rangle P_j.
\]

Thus we can conclude
3.4. LINEAR SYSTEMS WITH CONSTANT COEFFICIENTS

**Proposition 3.4.8.** If $A$ is an $n \times n$ matrix with a set of $n$ orthonormal eigenvectors $\{v_j\}$ and eigenvalues $\{\lambda_j\}$, then the solution of

$$y' = Ay, \quad y(0) = y_0$$

is

$$y = e^{At}y_0 = \sum_{j=1}^{n} e^{\lambda_j t} \langle y_0, v_j \rangle v_j.$$

For the nonhomogeneous problem

$$y' = Ay + B, \quad y(0) = y_0$$

the variation of parameters formula gives

$$y = e^{At}y_0 + \int_0^t e^{A(t-s)} B(s) \, ds$$

$$= \sum_{j=1}^{n} e^{\lambda_j t} \langle y_0, v_j \rangle v_j + \sum_{j=1}^{n} \langle y_0, v_j \rangle v_j \int_0^t e^{\lambda_j (t-s)} B(s) \, ds.$$

Unfortunately, as we know, even for the simple example $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, a matrix can be not diagonalizable. This means it does not have a complete set of linearly independent eigenfunctions. For this reason we need a more general procedure to compute $e^{At}$. One method is the following algorithm.\(^1\)

**Theorem 3.4.9.** Let $A$ be an $n \times n$ constant matrix. Let $P(\lambda) = \det(A - \lambda I)$ be the characteristic polynomial of $A$. Let $r_1, \ldots, r_n$ denote the solutions of the scalar constant coefficient linear equation $P(\lambda) = 0$ that satisfy the following initial conditions:

\[
\begin{align*}
\begin{cases}
  r_1(0) = 1 \\
r_1'(0) = 0 \\
  \vdots \\
r_1^{(n-1)}(0) = 0
\end{cases},
\begin{cases}
  r_2(0) = 0 \\
r_2'(0) = 1 \\
  \vdots \\
r_2^{(n-1)}(0) = 0
\end{cases},
\begin{cases}
  \vdots \\
  \vdots \\
  \vdots \\
  \vdots
\end{cases},
\begin{cases}
  r_n(0) = 0 \\
r_n'(0) = 0 \\
  \vdots \\
r_n^{(n-1)}(0) = 1
\end{cases}
\end{align*}
\]

Then,

$$e^{At} = r_1(t)I + r_2(t)A + r_3(t)A^2 + \cdots + r_n(t)A^{n-1}.$$  \hfill (3.4.5)

Before proving the theorem, we give some examples. First, let’s develop some computational techniques. The roots of the characteristic polynomial of $A$ are, of course, the eigenvalues of $A$. We only need to know the roots of the characteristic polynomial and their multiplicities to find the general solution of $P(D)r = 0$. So, suppose that $s_1, \ldots, s_n$ are a fundamental set of solutions for this equation. Any solution $r$ can be written in the form $r = c_1 s_1 + \cdots + c_n s_n$ for constants $c_1, \ldots, c_n$.

Formally, we could write this as

$$r = \begin{bmatrix} s_1 & s_2 & \cdots & s_n \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = Sc,$$

where $S$ is the row vector of fundamental solutions.

If we want $r$ to satisfy the initial conditions $r(0) = \gamma_1$, $r'(0) = \gamma_2$, $\ldots$, $r^{(n-1)}(0) = \gamma_n$, we need to find the $c_j$’s by solving the equations

$$c_1 s_1^{(j)}(0) + c_2 s_2^{(j)}(0) + \cdots + c_n s_n^{(j)}(0) = \gamma_j, \quad j = 0, \ldots, n - 1.$$ 

We can write this in matrix form as

$$Sc = \gamma$$

where $S$ is the matrix

$$\begin{bmatrix} s_1(0) & s_2(0) & \cdots & s_n(0) \\ s_1'(0) & s_2'(0) & \cdots & s_n'(0) \\ \vdots & \vdots & \ddots & \vdots \\ s_1^{(n-1)}(0) & s_2^{(n-1)}(0) & \cdots & s_n^{(n-1)}(0) \end{bmatrix}$$

i.e., the Wronskian matrix at 0. Of course, we are using the notation

$$c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}, \quad \gamma = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_n \end{bmatrix},$$

so $\gamma$ is the vector of initial conditions.

Suppose that we want to solve several initial value problems with with right hand side vectors $\beta_1, \ldots, \beta_k$. If $r_j$ is the solution of with initial conditions $\beta_j$, then $r_j = Sc_j$ where $c_j$ is the solution of $Sc_j = \beta_j$. If we introduce the matrices

$$C = \begin{bmatrix} c_1 & c_2 & \cdots & c_k \end{bmatrix}, \quad B = \begin{bmatrix} \beta_1 & \beta_2 & \cdots & \beta_k \end{bmatrix},$$
we can combine the equations $S c_j = \beta_j$ into the single matrix equation $SC = B$. The solution is $C = S^{-1}B$ Thus, we can summarize the solutions $r_j$ as

$$[r_1 \ r_2 \ \ldots \ r_n] = SC,$$

where $C = S^{-1}B$.

In the case of the collection of initial value problems (3.4.4), the right hand side vectors are the standard basis vectors $e_1, e_2, \ldots, e_n$ and so $B$ is the identity matrix. Thus, the solutions $r_1, \ldots, r_n$ required in the theorem are given by

$$[r_1 \ r_2 \ \ldots \ r_n] = 8S^{-1}.$$

**Example 3.4.10.** Construct $e^{At}$ and find the solution of $y' = Ay$, $y(0) = y_0$, where

$$A = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}, \quad y_0 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

The characteristic polynomial is $P(\lambda) = \det(A - \lambda) = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2$. Thus, 2 is a root of multiplicity two. To find the eigenvectors, we calculate the kernel of

$$A - 2I = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

This matrix is row equivalent to the matrix

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

Thus, the $\lambda = 2$ eigenspace is spanned by the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. It follows that $A$ is not diagonalizable.

To apply the algorithm, we need to solve the initial value problem $(D^2 - 4D + 4)r = 0$ for the two sets of initial conditions $r_1(0) = 1, r_1'(0) = 0$ and $r_2(0) = 0, r_2'(0) = 1$. A fundamental set of solutions for this equation is $S = [e^{2t}, te^{2t}]$. The general Wronskian matrix is

$$\begin{bmatrix} e^{2t} & te^{2t} \\ 2e^{2t} & e^{2t} + 2te^{2t} \end{bmatrix}$$

and the Wronskian matrix at 0 is

$$S = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}.$$
We compute that
\[ S^{-1} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}. \]

Thus, we have
\[ \begin{bmatrix} r_1 & r_2 \end{bmatrix} = 8S^{-1} = [e^{2t}, te^{2t}] \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \]
or, to put it another way,
\[ r_1 = e^{2t} - 2te^{2t}, \]
\[ r_2 = te^{2t}. \]

Thus, according to the theorem,
\[ e^{At} = r_1 I + r_2 A = (e^{2t} - 2te^{2t}) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + te^{2t} \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} e^{2t} + te^{2t} & -te^{2t} \\ te^{2t} & e^{2t} - te^{2t} \end{bmatrix}. \]

The solution of the initial value problem is
\[ y = e^{At} y_0, \]
or
\[ y = \begin{bmatrix} e^{2t} + te^{2t} & -te^{2t} \\ te^{2t} & e^{2t} - te^{2t} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2e^{2t} + te^{2t} \\ e^{2t} + e^{2t} \end{bmatrix}. \]

Example 3.4.11. Find \( e^{At} \), where
\[ A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & 2 & 2 \end{bmatrix}. \]

The characteristic polynomial is \( P(\lambda) = \det(A - \lambda I) = (\lambda - 1)^2(\lambda - 2) \). Thus, 1 is a root of multiplicity two and 2 is a root of multiplicity one. A fundamental set of solutions for the equation \( P(D)r = 0 \) is \( e^t, te^t, e^{2t} \). The general Wronskian matrix is
\[ \begin{bmatrix} e^t & te^t & e^{2t} \\ e^t (1 + t)e^t & 2e^{2t} \\ e^t (2 + t)e^t & 4e^{2t} \end{bmatrix}. \]
so the Wronskian matrix at $t = 0$ is

$$S = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix}.$$ 

Thus, we compute that

$$\begin{bmatrix} r_1 & r_2 & r_3 \end{bmatrix} = 8S^{-1}$$

$$= \begin{bmatrix} e^t & te^t & e^{2t} \\ 0 & 2 & -1 \\ -2 & 3 & -1 \\ 1 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2te^t + e^{2t} & (2 + 3t)e^t - 2e^{2t} & (-1 - t)e^t + e^{2t} \end{bmatrix}.$$ 

By the theorem,

$$e^{At} = (-2te^t + e^{2t})I + ((2 + 3t)e^t - 2e^{2t})A + ((-1 - t)e^t + e^{2t})A^2$$

$$= (-2te^t + e^{2t})\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + ((2 + 3t)e^t - 2e^{2t})\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & 2 & 2 \end{bmatrix}$$

$$+ ((-1 - t)e^t + e^{2t})\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ -3 & 5 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} e^t & te^t & 0 \\ 0 & e^t & 0 \\ e^t - e^{2t} & (-1 + t)e^t + e^{2t} & e^{2t} \end{bmatrix}.$$ 

Example 3.4.12. Find $e^{At}$, where

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$ 

The characteristic polynomial is $P(\lambda) = \lambda^3 - 5\lambda^2 + 9\lambda - 5 = (\lambda - 1)(\lambda^2 - 4\lambda + 5)$. The roots are $1, 2 + i, 2 - i$. Since $A$ is real, $e^{At}$ must be real. Thus, it makes sense to
use a real fundamental set of solutions for \( P(D)r = 0 \). A fundamental set of solutions is
\[
S = \begin{bmatrix}
    e^t & e^{2t} \cos(t) & e^{2t} \sin(t)
\end{bmatrix}.
\]
The Wronskian matrix of this at \( t = 0 \) is
\[
S = \begin{bmatrix}
    1 & 1 & 0 \\
    1 & 2 & 1 \\
    1 & 3 & 4
\end{bmatrix}.
\]
Thus, the vector of the \( r \)'s is
\[
\left[ \frac{5e^t}{2} - \frac{3e^{2t} \cos(t)}{2} + \frac{e^{2t} \sin(t)}{2} \quad -2e^t + 2e^{2t} \cos(t) - e^{2t} \sin(t) \quad e^t - \frac{e^{2t} \cos(t)}{2} + \frac{e^{2t} \sin(t)}{2} \right].
\]
Thus, by the theorem,
\[
e^{At} = r_1 I + r_2 A + r_3 A^2
\]
We now turn to the proof of the theorem. The basic ingredient is the Cayley-Hamilton theorem. Suppose that \( \hat{A} \) is an \( n \times n \) matrix and let \( P(\lambda) = \det(\hat{A} - \lambda I) \) be the characteristic polynomial of \( \hat{A} \). We can write
\[
P(\lambda) = \alpha_n \lambda^n + \alpha_{n-1} \lambda^{n-1} + \cdots + \alpha_1 \lambda + \alpha_0.
\]
it’s not hard to show that \( \alpha_n = (-1)^n \). It makes sense to plug \( \hat{A} \) into the polynomial \( P(\lambda) \), namely
\[
P(\hat{A}) = \alpha_n A^n + \cdots + \alpha_1 A + \alpha_0 I.
\]
The Cayley-Hamilton Theorem says that \( P(\hat{A}) = 0 \). A brief proof of the Cayley-Hamilton Theorem is given in Section 3.5.

The next ingredient of the proof of the theorem is the following lemma.

**Lemma 3.4.13.** Let \( A \) be an \( n \times n \) matrix (real or complex) and let \( P(\lambda) = \sum \alpha_j \lambda^j \) be the characteristic polynomial of \( A \).

Consider the \( n \)-order differential equation for an \( n \times n \) matrix valued function \( t \mapsto \Phi(t) \),
\[
\alpha_n \Phi^{(n)}(t) + \alpha_{n-1} \Phi^{(n-1)}(t) + \cdots + \alpha_1 \Phi'(t) + \alpha_0 \Phi(t) = 0.
\]

Then, the unique solution \( \Phi \) of (3.4.6) subject to the initial conditions
\[
\Phi(0) = I, \quad \Phi'(0) = A, \quad \Phi''(0) = A^2, \quad \ldots, \quad \Phi^{(n-1)}(0) = A^{n-1}
\]
is \( \Phi(t) = e^{At} \).
Proof. Consider the uniqueness of the solution first. So, suppose that $\Phi_1$ and $\Phi_2$ are solutions of (3.4.6) subject to the initial conditions (3.4.7). Then $\Psi = \Phi_2 - \Phi_1$ is a solution of (3.4.6) which satisfies the initial conditions

$$\Psi(0) = 0, \quad \Psi'(0) = 0, \quad \Psi''(0) = 0, \quad \ldots, \quad \Psi^{(n-1)}(0) = 0.$$ 

But this means that each entry $\psi_{ij}(t)$ of $\Psi(t)$ is a solution of the scalar $n$-order differential equation

$$\alpha_n y^{(n)}(t) + \alpha_{n-1} y^{(n-1)}(t) + \cdots + \alpha_1 y'(t) + \alpha_0 y(t) = P(D)y = 0$$

which satisfies the conditions

$$y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 0, \quad \ldots, \quad y^{(n-1)}(0) = 0.$$ 

Thus, each entry of $\Psi$ is identically zero. This shows that $\Phi_1 = \Phi_2$.

To show that $\Phi(t) = e^{At}$ is a solution, note that

$$\Phi^{(k)}(t) = A^k e^{At}. \quad (3.4.8)$$

Thus, $\Phi^{(k)}(0) = A^k$, so this function $\Phi$ satisfies the initial conditions (3.4.7). If we plug $\Phi$ into the left hand side of (3.4.6), using (3.4.8), we get

$$\alpha_n A^n e^{At} + \alpha_{n-1} A^{n-1} e^{At} + \cdots + \alpha_1 A e^{At} + \alpha_0 e^{At} = P(A)e^{At} = 0,$$

since $P(A) = 0$ by the Cayley-Hamilton Theorem. Thus, $\Phi(t) = e^{At}$ is a solution of (3.4.6) with initial conditions (3.4.7).

We can now complete the proof of Theorem 3.4.9. By the last Lemma, all we need to do is to show that the expression

$$\Phi(t) = \sum_{j=0}^{n-1} r_{j+1}(t) A^j$$

given in the theorem satisfies the differential equation (3.4.6) and the initial conditions (3.4.7). We have

$$\Phi^{(k)}(t) = \sum_{j=0}^{n-1} r^{(k)}_{j+1}(t) A^j. \quad (3.4.9)$$

Thus, we have

$$\Phi^{(k)}(0) = \sum_{j=0}^{n-1} r^{(k)}_{j+1}(0) A^j = A^k,$$
since the $r_j$’s satisfy the initial conditions (3.4.4). The differential equation is

$$\sum_{k=0}^{n} \alpha_k \Phi^{(k)}(t) = 0.$$  

Substituting (3.4.9) into the left hand side, we have

$$\sum_{k=0}^{n} \alpha_k \Phi^{(k)}(t) = \sum_{k=0}^{n} \alpha_k \sum_{j=0}^{n-1} r^{(k)}_{j+1}(t) A^j$$

$$= \sum_{j=0}^{n-1} \left[ \sum_{k=0}^{n} \alpha_k r^{(k)}_{j+1}(t) \right] A^j.$$  

But, each of the coefficients

$$\left[ \sum_{k=0}^{n} \alpha_k r^{(k)}_{j+1}(t) \right], \quad j = 0, \ldots, n - 1$$  

is zero, because each $r_j$ is a solution of the scalar equation

$$P(D)r = \sum_{k=0}^{n} \alpha_k r^{(k)} = 0.$$  

This completes the proof of the theorem.

In the next chapter we will establish some estimates on how fast $e^{At}$ grows as $t$ goes to infinity.
Exercises for Chapter 3

1. Prove Corollary 2.3.1 (in the notes). Use Thm. 2.3.4.

2. Prove the special case of Gronwall’s Inequality when \( p(t) = k \) and \( f_2(t) = \delta \) are constant. That is, show that in this case one gets that \( f_1(t) \leq \delta e^{k|x-a|} \).

3. Suppose the \( u, v \) are linearly independent and continuous on and interval \( I \). Suppose that \( w \) is defined on \( I \) and has only finitely many zeros. Show that \( uw, vw \) are linearly independent on \( I \). Show that the result fails if \( u, v \) are not continuous.

4. Show that the solution of the initial value problem

\[
ay'' + by' + cy = g(t), \quad y(t_0) = 0, y'(t_0) = 0
\]

where \( a, b, c \), are constants, has the form

\[
y = \phi(t) = \int_{t_0}^{t} K(t - s)g(s)ds.
\]

The function \( K \) depends only on the linearly independent solutions \( y_2, y_2 \) of the corresponding homogeneous equation and is independent of the inhomogeneous term \( g \). Note also that \( K \) depends only on the combination of \( t - s \) and hence is actually a function of a single variable. Think of \( g \) as the input and \( \phi(t) \) as the output. The result shows that the output depends on the input over the entire interval from the initial time \( t_0 \) to the current time \( t \). The integral is called the convolution of \( K \) and \( g \).

5. Find the general solution.

   a) \( y'' + y = \sin x \sin 2x \).

   b) \( y'' - 4y' + 4y = e^x + e^{2x} \).

   c) \( y''' + y'' + y' = 1 + \cos(\frac{\sqrt{3}x}{2}) \).

6. Find \( e^{At} \) for

   a) \( A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \)  
   b) \( A = \begin{bmatrix} 0 & 1 \\ 8 & -2 \end{bmatrix} \)  
   c) \( A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \)

7. Find \( e^{At} \) for
a) \( A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \)  

b) \( A = \begin{bmatrix} 4 & 5 \\ -4 & -4 \end{bmatrix} \)  
c) \( A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \)

d) \( A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \)  
e) \( A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \)

8. Solve the equation \( \dot{x} = Ax + B \) with \( x(0) = x_0 \)

(a) \( A = \begin{bmatrix} 0 & 1 \\ 8 & -2 \end{bmatrix} \) and \( B = 0, \ x_0 = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \)

(b) \( A = \begin{bmatrix} 0 & 1 \\ -9 & 6 \end{bmatrix} \) and \( B = \begin{bmatrix} 0 \\ t \end{bmatrix}, \ x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \)

c) \( A = \begin{bmatrix} 0 & 1 \\ 8 & -2 \end{bmatrix} \) and \( B = \begin{bmatrix} 0 \\ 4 \end{bmatrix}, \ x_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \)
3.5 Appendix: The Cayley-Hamilton Theorem

Some readers will not have seen a proof of the Cayley-Hamilton Theorem, and many references where you might look it up go farther into the structure theory of linear transformations than is necessary for us before they get to the Cayley-Hamilton Theorem. In this section, we give a concise, self-contained, proof of the Cayley-Hamilton Theorem.

It is convenient to approach the problem in terms of linear transformations. Suppose that \( V \) is a vector space over \( \mathbb{K} (= \mathbb{R} \text{ or } \mathbb{C}) \). Let \( T: V \to V \) be a linear transformation.

If \( v_1, \ldots, v_n \) is an ordered basis of \( V \), we can find a matrix \( A = [a_{ij}] \) that represents \( T \) with respect to this basis. The matrix \( A \) can be described as

\[
T(v_j) = \sum_{i=1}^{n} a_{ij} v_i, \quad j = 1, \ldots, n,
\]

i.e., the \( j \)th column of \( A \) gives the coefficients of \( T(v_j) \) with respect to our given basis.

Suppose that \( u_1, \ldots, u_n \) is another basis, and let \( B \) be the matrix of \( T \) with respect to this basis. There is a nonsingular matrix \( P \) such that

\[
u_j = \sum_{i=1}^{n} p_{ij} v_i, \quad j = 1, \ldots, n.
\]

It is then pretty easy to calculate that

\[
B = P^{-1}AP.
\]

Now, note that

\[
det(B) = det(P^{-1}AP) = det(P^{-1}) \det(A) \det(P) = det(P)^{-1} \det(A) \det(P) = \det(A).
\]

Thus, we may define \( \det(T) \) by \( \det(T) = \det(A) \), where \( A \) is the matrix representation of \( T \) with respect to some basis—we get the same number no matter which basis we use.

There are standard algebraic operations defined on the set \( L(V) \) of linear transformations \( V \to V \), namely

\[
(ST)(v) = S(T(v))
\]
\[
(S + T)(v) = S(v) + T(v)
\]
\[
(kT)(v) = kT(v), \quad k \in \mathbb{K}.
\]

If \( Q(\lambda) = a_n \lambda^n + \ldots + a_1 \lambda + a_0 \) is a polynomial with coefficients in \( \mathbb{K} \), we define

\[
Q(T) = a_n T^n + a_{n-1} T^{n-1} + \cdots + a_1 T + a_0 I,
\]
where \( I \) is the identity linear transformation on \( V \). The algebraic operations on matrices are defined precisely to correspond to the algebraic operations on linear transformations. Thus, if we choose a basis for \( V \) and \( A \) is the matrix of \( T \) with respect to this basis, the matrix of \( Q(T) \) with respect to this basis is

\[
Q(A) = a_n A^n + a_{n-1} A^{n-1} + \cdots + a_1 A + a_0 I,
\]

where \( I \) is the identity matrix.

Suppose that \( V \) is a complex vector space and \( T: V \to V \) is a linear transformation. The function \( P_T: \mathbb{C} \to \mathbb{C} \) defined by \( P_T(z) = \det(T - zI) \) makes sense. If we choose a basis of \( V \) and let \( A \) be the matrix of \( T \) with respect to this basis, we have \( P_T(z) = \det(A - zI) \). Thus, \( P_T(z) \) is a polynomial, which we call the characteristic polynomial of \( T \). We see that \( P_T(z) \) is the same as the characteristic polynomial \( P_A(z) = \det(A - zI) \) of any matrix representation \( A \) of \( T \). If we show that \( P_T(T) = 0 \), it will follow that \( P_A(A) = 0 \) for every matrix representation \( A \) of \( T \) (since \( P_A(A) \) is the matrix representation of \( P_T(T) \)).

On the other hand, every matrix \( A \) with complex entries (which includes the case of real entries) is the matrix representation of some linear transformation on a complex vector space, for example, \( T: \mathbb{C}^n \to \mathbb{C}^n; v \mapsto Av \). Thus, if we show that \( P_T(T) = 0 \) for every linear transformation, it will follow that \( P_A(A) = 0 \) for every matrix \( A \).

Thus, it will suffice to prove the following version of the Cayley-Hamilton Theorem.

**Theorem 3.5.1 (Cayley-Hamilton).** Let \( V \) be a finite dimensional vector space over \( \mathbb{C} \) and let \( T: V \to V \) be a linear transformation. Then, if \( P(\lambda) = \det(T - \lambda I) \) is the characteristic polynomial of \( T \), \( P(T) = 0 \).

The proof will occupy the remainder of this section. To begin, let \( n \) be the dimension of \( V \). We make the following definition: a fan for \( T \) is a collection \( \{V_j\}; j = 1^n \) of subspaces of \( V \) with the following properties

\[
V_1 \subset V_2 \subset \cdots \subset V_{n-1} \subset V_n = V,
\]

\[
\dim(V_j) = j, \quad T(V_j) \subset V_j, \quad j = 1, \ldots, n.
\]

The main step in the proof is the following lemma.

**Lemma 3.5.2.** If \( T: V \to V \) is a linear transformation on a finite dimensional complex vector space \( V \), there exists a fan for \( T \).

**Proof of Lemma.** The proof is by induction on the dimension of \( V \). The lemma is trivially true in the case where the dimension of \( V \) is one (the fan is just \( \{V\} \)).

So, suppose that the lemma is true when the dimension of the vector space is \( n - 1 \). Assume that \( T: V \to V \), where \( V \) has dimension \( n \).
Since we are working over the complex numbers, $T$ must have an eigenvector (the characteristic polynomial has at least one complex root). Thus, there is a non-zero vector $v_1$ and a complex number $\lambda$ such that $T(v_1) = \lambda v_1$. Let $V_1$ be the one dimensional subspace of $V$ spanned by $v_1$.

We can find a subspace $W \subset V$ of dimension $n - 1$ such that

$$V = V_1 \oplus W. \quad (3.5.10)$$

One way to see this is to note that the set $\{v_1\}$ is linearly independent. Any linearly independent set can be completed to a basis, so we can find vectors $\{w_j\}_{j=1}^{n-1}$ such that $v_1, w_1, \ldots, w_{n-1}$ is a basis of the $n$-dimensional space $V$. We can then take $W$ to be the span of $\{w_j\}_{j=1}^{n-1}$. The direct sum decomposition (3.5.10) means that every $v \in V$ can be written as $v = \zeta v_1 + w$ for a unique scalar $\zeta \in \mathbb{C}$ and vector $w \in W$.

Let $P: V \to V$ and $Q: V \to V$ be the projections onto $V_1$ and $W$ respectively. In other words, if $v \in V$, write $v = \zeta v_1 + w$, where $w \in W$. Then

$$P(v) = P(\zeta v_1 + w) = \zeta v_1$$
$$Q(v) = Q(\zeta v_1 + w) = w.$$

It’s easy to check that $P$ and $Q$ are linear transformations, and clearly $v = P(v) + Q(v)$.

Since $Q(V) \subset W$, surely $QT(W) \subset W$. Thus, we may restrict $QT$ to $W$ and get a linear transformation $W \to W$. By the induction hypothesis, there is a fan for this linear transformation. Thus, we have subspaces

$$W_1 \subset W_2 \subset \cdots \subset W_{n-1} = W$$

such that $\dim(W_k) = k$ and $QT(W_k) \subset W_k$. Now define subspaces $V_2, \ldots, V_n$ by

$$V_1 = V_1, \quad (V_1 \text{ is already defined})$$
$$V_2 = V_1 + W_1$$
$$V_3 = V_1 + W_2$$
$$\vdots$$
$$V_n = V_1 + W_{n-1}.$$

Thus, $V_j = V_1 + W_{j-1}$ for $j = 2, \ldots, n$ and $V_n = V_1 + W = V$. From the direct sum decomposition (3.5.10), it is clear that these sums are direct and so $\dim(V_j) = j$. To prove that $\{V_j\}$ is a fan for $T$, it remains to prove that $T(V_j) \subset V_j$ for each $j$.

In the case of $V_1$, this is clear because $v_1$ is an eigenvector of $T$. An arbitrary element of $V_1$ is of the form $\zeta v_1$. Then $T(\zeta v_1) = \lambda \zeta v_1 \in V_1$. 

For $j > 1$, a typical element $v$ of $V_j$ can be written as $v = \zeta v_1 + w$, where $w \in W_{j-1}$. We need to show that $T(v) \in V_j$. We have $T(v) = PT(v) + QT(v)$. Certainly $PT(v) \in V_1 \subset V_j$, since $P$ is projection onto $V_1$. For $QT(v)$, we have

\[
QT(v) = QT(\zeta v_1 + w) \\
= Q(\zeta T(v_1) + T(w)) \\
= \zeta \lambda Q(v_1) + QT(w) \\
= QT(w),
\]

since $Q(v_1) = 0$. But $W_{j-1}$ is invariant under $QT$, so $QT(v) = QT(w) \in W_{j-1} \subset V_j$. Thus, $V_j$ is invariant under $T$. 

To proceed with the proof of the theorem, suppose that $T: V \to V$ where $V$ has dimension $n$ and let $\{V_j\}$ be a fan for $T$.

We construct a basis of $V$ as follows. Choose any non-zero $v_1$ in the one dimensional space $V_1$. Obviously $V_1 = \text{span}\ \{v_1\}$.

Now, since $V_1 \subset V_2$ and the dimension of $V_2$ is 2 (which is greater than the dimension of $V_1$), we can find a vector $v_2 \in V_2$ that is not in $V_1$. We claim that the collection $v_1, v_2$ is linearly independent. To see this, suppose that $c_1 v_1 + c_2 v_2 = 0$. If $c_2 = 0$, then $c_1$ must be zero because $v_1 \neq 0$. On the other hand, if we had $c_2 \neq 0$, we would have $v_2 = (-c_1/c_2) v_1$, which implies that $v_2$ is in the subspace $V_1$. This contradiction shows that we must have $c_2 = 0$, and so also $c_1 = 0$. Since $v_1, v_2 \in V_2$ and $V_2$ has dimension 2, we must have $\text{span}\ \{v_1, v_2\} = V_2$

Suppose that we have constructed linearly independent vectors $\{v_j\}_{j=1}^{k}$, where $k < n$, such that

$$V_j = \text{span}\ \{v_1, \ldots, v_j\}, \quad j = 1, \ldots, k.$$  

To extend this set, choose a vector $v_{k+1} \in V_{k+1} \setminus V_k$. We claim that $v_1, \ldots, v_k, v_{k+1}$ is linearly independent. To see this, suppose that

$$c_1 v_1 + \cdots + c_k v_k + c_{k+1} v_{k+1} = 0.$$  

If $c_{k+1} = 0$, we must have $c_1 = 0, \ldots, c_k = 0$, since we have assumed that $v_1, \ldots, v_k$ are linearly independent. But, if we had $c_{k+1} \neq 0$, we could write

$$v_{k+1} = (-c_1/c_{k+1}) v_1 - \cdots - (c_k/c_{k+1}) v_k.$$  

This implies that $v_{k+1} \in \text{span}\ \{v_1, \ldots, v_k\} = V_k$, which contradicts the choice of $v_{k+1}$. Thus, we conclude that all of the $c_j$'s are zero, and so that $v_1, \ldots, v_{k+1}$ is linearly independent. Since $\text{span}\ \{v_1, \ldots, v_{k+1}\} \subset V_{k+1}$ and the dimensions are equal, we must have $V_{k+1} = \text{span}\ \{v_1, \ldots, v_{k+1}\}$.
Continuing in the way, we get a basis \( v_1, \ldots, v_n \) of \( V \) such that

\[
V_k = \text{span} \{ v_1, \ldots, v_k \}, \quad k = 1, \ldots, n.
\]

Let’s see what the matrix of \( T \) is with respect to this basis. Since \( v_1 \in V_1 \) and \( V_1 \) is invariant under \( T \), we must have \( T(v_1) \in V_1 = \text{span} \{ v_1 \} \). Thus, the expansion of \( T(v_1) \) in terms of the basis is

\[
T(v_1) = a_{11} v_1
\]

for a scalar \( a_{11} \). In general, we have \( v_k \in V_k \) and \( V_k \) is invariant under \( T \), so \( T(v_k) \in V_k = \text{span} \{ v_1, \ldots, v_k \} \), and so the expansion of \( T(v_k) \) is

\[
T(v_k) = a_{1k} v_1 + a_{2k} v_2 + \cdots + a_{kk} v_k.
\]

Thus, the matrix \( A \) of \( T \) with respect to this basis is of the form

\[
\begin{bmatrix}
  a_{11} & a_{12} & a_{13} & \cdots & a_{1,n-1} & a_{1n} \\
  0 & a_{22} & a_{23} & \cdots & a_{2,n-1} & a_{2n} \\
  0 & 0 & a_{33} & \cdots & a_{3,n-1} & a_{3n} \\
  0 & 0 & 0 & \cdots & a_{4,n-1} & a_{4n} \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & 0 & \cdots & a_{n-1,n-1} & a_{n-1,n} \\
  0 & 0 & 0 & \cdots & 0 & a_{nn}
\end{bmatrix}
\]

In other words, the matrix is upper triangular. Since it’s easy to compute the determinant of an upper triangular matrix, we see that the characteristic polynomial of \( T \) (which is the same as the characteristic polynomial of \( A \)) is given by

\[
P(\lambda) = (a_{11} - \lambda)(a_{22} - \lambda) \cdots (a_{nn} - \lambda)
\]

and so

\[
P(T) = (a_{11}I - T)(a_{22}I - T) \cdots (a_{nn}I - T).
\]

To complete the proof, we will show by induction on \( k \) that

\[
(a_{11}I - T) \cdots (a_{kk}I - T)V_k = 0
\]

for \( k = 1, \ldots, n \). The case \( k = n \) then shows that \( P(T)V_n = P(T)V = 0 \), so \( P(T) = 0 \).

For \( k = 1 \), we see that \( a_{11}I - T \) is zero on \( V_1 \), because every vector in \( V_1 \) is an eigenvector of \( T \) with eigenvalue \( a_{11} \).
For the induction step, assume that (3.5.12) holds for some \( k < n \). An arbitrary vector \( v \in V_{k+1} \) can be written as \( v = c_1v_1 + \cdots + c_kv_k + \alpha v_{k+1} + u \), where \( u = c_1v_1 + \cdots + c_kv_k \in V_k \). Then, we have

\[
T(v) = \alpha T(v_{k+1}) + T(u).
\]

Since \( V_k \) is invariant under \( T \), \( T(u) \) is some vector in \( V_k \), call it \( u' \). If we apply (3.5.11), we have

\[
T(v) = \alpha a_{k+1,k+1}v_{k+1} + \left[ \alpha \sum_{j=1}^{k} a_{j,k+1}v_j \right] + u'.
\]

The term in brackets is in \( V_k = \text{span} \{v_1, \ldots, v_k\} \) and \( u' \in V_k \), so we have

\[
T(v) = \alpha a_{k+1,k+1}v_{k+1} + u'',
\]

where \( u'' \in V_k \). Thus, we have

\[
a_{k+1,k+1}v - T(v) = a_{k+1,k+1}\alpha v_{k+1} + a_{k+1,k+1}u - \alpha a_{k+1,k+1}v_{k+1} - u''
= a_{k+1,k+1}u - u'' \in V_k.
\]

Thus, we have shown that \((a_{k+1,k+1}I - T)V_{k+1} \subset V_k\). But then

\[
(a_{11}I - T) \cdots (a_{kk}I - T)(a_{k+1,k+1} - T)V_{k+1} \subset (a_{11}I - T) \cdots (a_{kk}I - T)V_k
\]

and the right hand side is 0 by the induction hypothesis (3.5.12).

This completes the proof of the Cayley-Hamilton Theorem.