Output Regulation for Linear Distributed Parameter Systems

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Abstract—This work extends the geometric theory of output regulation to linear distributed parameter systems with bounded input and output operator, in the case when the reference signal and disturbances are generated by a finite dimensional exogenous system. In particular, it is shown that the full state feedback and error feedback regulator problems are solvable, under the standard assumptions of stabilizability and detectability, if and only if a pair of regulator equations is solvable. For linear distributed parameter systems this represents an extension of the geometric theory of output regulation developed in [10] and [4]. We also provide simple criteria for solvability of the regulator equations based on the eigenvalues of the exosystem and the system transfer function. Examples are given of periodic tracking, set point control, and disturbance attenuation for parabolic systems and periodic tracking for a damped hyperbolic system.

Index Terms—Bounded input and output operators, distributed parameter systems, regulator problem.

I. INTRODUCTION

ONE of the central problems in control theory is to control a fixed plant so that its output tracks a reference signal (and/or rejects a disturbance) produced by an external generator or exogenous system. Generally two versions of this problem are considered. In the first, the state feedback regulator problem, the controller is provided with full information of the state of the plant and exosystem. For the second, and perhaps more realistic error feedback regulator problem, only the components of the error are available for measurement. For linear finite-dimensional systems it has been shown by Francis [10] that the solvability of the regulator problem is equivalent to the solvability of a pair of linear matrix Sylvester equations. This in turn can be characterized as a property of the transmission polynomials of the composite system formed from the plant and the exosystem, as was shown by Hautus [12]. Francis and Wonham [11] have also shown that any regulator that solves the error feedback problem has to incorporate a model of the exogenous system generating the reference signal which is to be tracked and/or the disturbance that must be rejected. This property is known as the internal model principle.

Similar results have been established for finite dimensional nonlinear systems in [4] in case the plant is exponentially stabilizable and the exosystem has bounded trajectories that do not trivially converge to zero. In particular, it is shown in [4] that the solvability conditions given by Francis in the linear case can be naturally generalized to the solvability of a pair of nonlinear equations, the regulator equations. Geometrically, these nonlinear regulator equations express the existence of a local manifold on which the actual and reference outputs coincide and which can be rendered invariant using feedback.

In this paper we develop the geometric methods introduced in [10] and [4] for solving the state and output feedback regulator problems for infinite-dimensional linear control systems, assuming that the control and observation operators are bounded. We expect to have more to say about the unbounded case in future papers. In particular we derive the regulator equations for a class of distributed parameters systems, obtaining an operator Sylvester equation. We also obtain results characterizing the solvability of both state and error feedback regulator problems in terms of solvability of these regulator equations. The main difficulties that arise in developing a geometric theory for the distributed parameter case are obvious: the phase space is infinite dimensional; the state operator is typically unbounded and consequently only densely defined; the error zeroing controlled invariant subspace must be contained in the domain of the state operator and the resulting regulator equations may become distributed parameter equations. Section II contains a formulation of the state and error feedback regulator problems and a discussion of the basic assumptions. In Section III, we present a motivating example of periodic tracking for a controlled heat equation. In Section IV we present our main solvability results in Theorems IV.1 and IV.2. In particular, we give necessary and sufficient conditions for the solvability of both the state and error feedback regulator problems in terms of the solvability of a pair of linear operator equations, the regulator equations. Once a solution of these equations is available, the appropriate state feedback control in the case of the full information problem, or a dynamic controller in the case of error feedback, are obtained which provide a solution to the regulator problem.

In Section V, we recall the definitions of transmission and invariant zeros. For certain classes of distributed parameter systems we provide a sequence of results, under various hypotheses on the plant and exosystem, expressing necessary and sufficient conditions for solvability of the regulator equations in terms of nonresonance conditions involving the eigenvalues of the exosystem and the plants transmission (or invariant) zeros.

Section VI contains several explicit examples of output regulation. In the first example we consider a problem of set point...
control for the heat equation in the presence of a periodic disturbance. The second example is related to the motivating example given in Section III. In this example we consider the problem of periodic tracking for the heat equation using error feedback. In the third example we consider a problem of periodic tracking for a damped wave equation. For each of these examples the regulator equations reduce to a system of linear ordinary differential equations subject to extra constraints. These systems can, in general, be readily solved numerically (or analytically in some cases) off-line to obtain approximate feedback controls that work very well in practice.

II. STATEMENT OF THE BASIC PROBLEMS

Consider a plant described by an abstract distributed parameter control system in Hilbert space

\[ \dot{z}(t) = Az(t) + Bu(t) + D(t) \]  
\[ y(t) = Cz(t) \quad \text{(measured output)} \]  
\[ z(0) = z_0 \]  

where

- \( z \in Z \) is the state of the system,
- \( Z \) is a complex separable Hilbert space (state space),
- \( u \in U \) is an input, and
- \( y \in Y \) is the measured output.

\( U \) and \( Y \) are real or complex separable Hilbert control and output spaces, respectively. \( A \) is assumed to be the infinitesimal generator of a strongly continuous semigroup on \( Z, B \in \mathcal{L}(U,Z) \) and \( C \in \mathcal{L}(Z,Y) \). The symbol \( \mathcal{L}(W_1,W_2) \) denotes the set of all bounded linear operators from a Hilbert space \( W_1 \) to a Hilbert space \( W_2 \). The term \( D(t) \) represents a disturbance.

In addition we will assume that there exists a finite dimensional linear system, referred to as the exogenous system (or exosystem), that produces a reference output \( y_r(t) \) and which is also used to model the disturbance \( D(t) \)

\[ \hat{w}(t) = Sw(t) \]  
\[ y_r(t) = Qw(t) \]  
\[ D(t) = Pw(t) \]  

Here \( S \in \mathcal{L}(W), Q \in \mathcal{L}(W,Y), \) and \( P \in \mathcal{L}(W,Z) \).

We denote the error between the measured and reference outputs by

\[ e(t) = y(t) - y_r(t) = Cz(t) - Qw(t). \]  

Problem II.1—State Feedback Regulator Problem: Find a feedback control law in the form

\[ u(t) = Kz(t) + Lw(t) \]

such that \( K \in \mathcal{L}(Z,U), L \in \mathcal{L}(W,U), \) and:

- (1a) the system \( \dot{z}(t) = (A + BK)z(t) \) is stable, i.e., \( A + BK \) is the infinitesimal generator of an exponentially stable \( C_0 \) semigroup;
- (1b) for the closed-loop system

\[ \dot{z}(t) = (A + BK)z(t) + (BL + P)w(t) \]  
\[ \hat{w}(t) = Sw(t) \]  

the error \( e(t) = Cz(t) - Qw(t) \) \( \to 0 \) as \( t \to \infty \), for any initial conditions \( z_0 \in Z \) in (II.3) and \( w_0 \in W \) in (II.4).

Problem II.2—Error Feedback Regulator Problem: Find an error feedback controller of the form

\[ \dot{X}(t) = FX(t) + Ge(t) \]  
\[ u(t) = HX(t) \]

where \( X(t) \in \mathcal{X} \) for \( t \geq 0, \mathcal{X} \) is a Hilbert space, \( G \in \mathcal{L}(Y, \mathcal{X}), H \in \mathcal{L}(\mathcal{X}, U) \) and \( F \) is the infinitesimal generator of a \( C_0 \) semigroup on \( \mathcal{X} \) with the properties that:

- (2a) the system

\[ \dot{z}(t) = Az(t) + BHX(t) \]  
\[ \dot{X}(t) = FX(t) + Ge(t) \]

is exponentially stable when \( u \equiv 0 \), i.e.

\[
\begin{bmatrix}
A & BH \\
GC & F
\end{bmatrix}
\]

is the infinitesimal generator of an exponentially stable \( C_0 \) semigroup;

- (2b) for the closed-loop system

\[ \dot{z}(t) = Az(t) + BHX(t) + Pw(t) \]  
\[ \dot{X}(t) = GCz(t) + FX(t) - GQw(t) \]  
\[ \hat{w}(t) = Sw(t) \]

the error \( e(t) = Cz(t) - Qw(t) \) \( \to 0 \) as \( t \to \infty \), for any initial conditions \( z_0 \in Z \) in (II.3), \( X(0) \in \mathcal{X} \) and \( w_0 \in W \) in (II.4).

As in [4], we impose the following standard assumptions.

Assumption II.1:

H1) The exosystem is neutrally stable, as in [4]. In the linear case, this is equivalent to the origin being Lyapunov stable forward and backward in time. This implies that \( \sigma(S) \subset \mathbb{j}\mathbb{R} \) (the imaginary axis) and \( S \) has no non-trivial Jordan blocks. Here and below we use the notation \( \sigma(T) \) for the spectrum of an operator \( T \). Also, by \( \rho(T) \) we will denote the resolvent set of \( T \).

H2) The pair \((A, B)\) is exponentially stabilizable, i.e., there exists \( K \in \mathcal{L}(Z, U) \) such that \( A + BK \) is the infinitesimal generator of an exponentially stable \( C_0 \) semigroup.

H3) The pair

\[
\begin{bmatrix}
A & P \\
0 & S
\end{bmatrix}, [C & -Q]
\]

is exponentially detectable, i.e., there exists \( G \in \mathcal{L}(Y, Z \times W) \), with

\[
G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}, \quad G_1 \in \mathcal{L}(Y,Z), G_2 \in \mathcal{L}(Y,W)
\]

such that

\[
\begin{bmatrix}
A & P \\
0 & S
\end{bmatrix} - C[G & -Q]
\]

is the infinitesimal generator of an exponentially stable \( C_0 \) semigroup.
These assumptions correspond to the standard hypotheses on which the finite-dimensional linear regulator theory is based (see, for example, [4, pp. 132–133] and [10]). The first of these, $H_1$, without loss of generality, excludes eigenvalues in the open left half-plane, since these trajectories decay exponentially to zero, and, asymptotically do not affect the output regulation. We remark that it is possible to prove all results in this work under the more general assumption that $S$ is an arbitrary $k \times k$ matrix with $\sigma(S) \subseteq \mathbb{C}^+$, i.e., that $S$ may have right half-plane eigenvalues and there may be nontrivial Jordan blocks. However, the proofs are more tedious and for the sake of brevity we shall present the main results for neutrally stable exosystems.

It is evident from the formulation of the state feedback problem, that for its solvability $H_2$ is a necessary condition. For finite-dimensional linear systems it is known that the stabilizability of $(A, B)$ and the detectability of $(C, A)$ are necessary for the solvability of the error feedback problem. The proof of this result first appeared in [10]. For distributed parameter systems the proof given in [10] has been extended to the present case in [14], provided that we make additional assumptions on the system (II.1). Assumption $H_3$ is a stronger condition than the exponential detectability of the pair $(C, A)$. Francis [10] showed that for the finite-dimensional linear error feedback problem this condition does not involve loss of generality in the following sense: the undetectability of the pair $\left(\begin{bmatrix} A & P \\ 0 & S \end{bmatrix}, [C \quad -Q]\right)$

indicates a redundancy in the exosystem and by eliminating superfluous exosystem variables it is possible to achieve $H_3$. This last property also generalizes to infinite-dimensional systems (cf., [14]), at least under the additional conditions imposed on the system to establish the necessity of the exponential stabilizability of $(A, B)$ and the exponential detectability of $(C, A)$ for the solvability of the error feedback problem. The argument is similar to the one given in [10] and will not be included.

III. MOTIVATING EXAMPLE: PERIODIC TRACKING FOR THE HEAT EQUATION

Consider a controlled one dimensional heat equation on the interval $[0, 1]$ with Neumann boundary conditions (cf. Curtain–Zwart [9])

$$\frac{\partial z}{\partial t}(x, t) = \frac{\partial^2 z}{\partial x^2}(x, t) + Bu(t)$$

(III.1)

$$\frac{\partial z}{\partial t}(0, t) = 0, \quad \frac{\partial z}{\partial t}(1, t) = 0$$

(III.2)

$$z(x, 0) = \phi(x), \quad y(t) = Cz(t).$$

(III.3)

We formulate the system (III.1), (III.3) in the form (II.1), (II.2) by choosing $Z = L^2(0, 1)$ and defining $A = \frac{d^2}{dx^2}$ with Neumann boundary conditions. In this case $A$ is an unbounded densely defined selfadjoint operator in $Z$

$$A\varphi = \varphi'', \quad \mathcal{D}(A) = \{\varphi \in H^2(0, 1); \varphi'(0) = \varphi'(1) = 0\}.$$

As it is well known the spectrum of $A$ is purely discrete

$$\sigma(A) = \{-k^2\pi^2\}_{k=0}^{\infty}$$

with corresponding orthonormal eigenvectors

$$\psi_0(x) = 1 \quad \text{and} \quad \psi_k(x) = \sqrt{2}\cos(k\pi x) \quad \text{for} \quad k \geq 1.$$

The operator $A$ generates a strongly continuous (in fact, analytic) semigroup on $Z$ given in terms of the orthonormal expansion

$$e^{At} \varphi = \sum_{j=0}^{\infty} e^{\lambda_j t} \langle \varphi, \psi_j \rangle \psi_j.$$ 

In this example, we consider a single-input/single-output system with bounded input and output operators $B$ and $C$ so that $Y = U = \mathbb{R}$.

1) Input Operator: The temperature input is spatially uniform over a small interval about a fixed point $x_0 \in (0, 1)$

$$Bu(t) = b(x_0)u$$

$$b(x) = \frac{1}{2\pi^2} \chi_{[x_0-\varepsilon, x_0+\varepsilon]}(x).$$

(III.4)

Here $\chi_{[a,b]}(x)$ denotes the characteristic function of the interval $[a, b]$.

$$\chi_{[a,b]}(x) = \begin{cases} 1, & x \in [a, b] \\ 0, & \text{otherwise} \end{cases}.$$

$B$ is a bounded linear operator defined by multiplication with the function $b \in Z$.

2) Output Operator: The output corresponds to the average temperature over a small interval about a point $x_1 \in (0, 1)$

$$C\varphi = \int_0^1 c(x)\varphi(x) \, dx$$

$$c(x) = \frac{1}{2\pi^2} \chi_{[x_1-\varepsilon, x_1+\varepsilon]}(x).$$

(III.5)

Since $C\varphi = \langle \varphi, c \rangle$ one can see that $C$ is a bounded linear observation functional on $Z$.

For simplicity, we assume that there are no disturbances, i.e., $d(t) = 0$, and that our design objective is to construct a control $u$ that will force the output $y(t)$ to track a periodic reference trajectory of the form $y_r(t) = M \sin(\omega t)$. In this case we may take the exogenous system in (II.4) to be a harmonic oscillator

$$\dot{\psi} = Sw, \quad S = \begin{bmatrix} 0 & \alpha \\ -\alpha & 0 \end{bmatrix}, \quad w(0) = \begin{bmatrix} 0 \\ M \end{bmatrix}.$$

In terms of our earlier notation, $Q = [1, 0]$, $P = 0$, and $S \in L(\mathbb{W})$ with $k = 2$ and $\mathbb{W} = \mathbb{R}^2$.

In our specific numerical example, we have chosen a noncollocated actuator and sensor pair with $x_0 = 3/4, x_1 = 1/4, \nu_0 = \nu_1 = 1/4$. We note that this system does not have “relative degree” one since $\langle \delta, \psi \rangle = 0$. We have set $M = 1$ and $\alpha = 2$ so that our reference signal is $y_r = \sin(2t)$. Finally, for our numerical simulations we have chosen the initial condition $\psi(x) = 4x^2(3/2 - x)$.

As a first attempt one might consider driving the system with the desired output, i.e., setting $u = M \sin(2t)$. Thus in our first simulation we have set the control input $u = \sin(2t)$. Fig. 1 contains a plot of the resulting outputs $y$ and $y_r$ while Fig. 2 depicts the error $\varepsilon(t) = y(t) - y_r(t)$.

As might be expected, due to the zero eigenvalue which causes the original uncontrolled system not to be asymptot-
Fig. 1. $y$ and $y_r$.

Fig. 2. $e(t) = y(t) - y_r(t)$.

Fig. 3. $y$ and $y_r$.

Fig. 4. $|e(t)|$.

Fig. 5. $e(t)$, $y(t)$, and $y_r(t)$.

Fig. 6. $|e(t)|$.
and $\Gamma = [\gamma_1, \gamma_2] \in \mathcal{L}(W, U)$ are solutions of the so-called regulator equations [see (IV.1) and (IV.2)]. Here these equations take the form

\[
\Pi S = A\Pi + B\Gamma \\
C\Pi = Q
\]

and are satisfied on the vector space $W$.

In this example the first regulator equation reduces to the following coupled system of second-order ordinary differential equations with boundary conditions:

\[
\Pi_1'(x) + \alpha \Pi_2(x) = -\gamma_1 b(x) \tag{III.8} \\
\Pi_2'(x) - \alpha \Pi_1(x) = -\gamma_2 b(x) \tag{III.9} \\
\Pi_j'(0) = \Pi_j'(1) = 0, \quad j = 1, 2. \tag{III.10}
\]

Conditions (III.10) correspond to the requirements that $\Pi_1$ and $\Pi_2$ lie in the domain of $A$.

The parameters $\gamma_1$ and $\gamma_2$ are chosen to satisfy the second regulator equation, which in this case reduces to the additional constraints

\[
\langle c, \Pi_1 \rangle = 1, \quad \langle c, \Pi_2 \rangle = 0. \tag{III.11}
\]

For this example it is possible to solve these equations explicitly using elementary techniques from the theory of ordinary differential equations. In particular, multiplying (III.9) by $j = \sqrt{-1}$ and adding the result to (III.8) we have

\[
(j\alpha I - A)\Pi_1 + j(j\alpha I - A)\Pi_2 = (\gamma_1 + j\gamma_2)b.
\]

Recall that $\alpha \neq 0$ and $\pm j\alpha \in \rho(A)$ so we can write this equation in the form

\[
\Pi_1 + j\Pi_2 = (\gamma_1 + j\gamma_2)(j\alpha I - A)^{-1}b. \tag{III.12}
\]

At this point we introduce the notation

\[
R_1 = \Re(j\alpha I - A)^{-1}b, \quad R_2 = \Im(j\alpha I - A)^{-1}b
\]

for the Real ($\Re$) and Imaginary ($\Im$) parts of $(j\alpha I - A)^{-1}b$. Then we can write (III.12), in terms of real and imaginary parts, as

\[
\Pi_1 + j\Pi_2 = (\gamma_1 R_1 - \gamma_2 R_2) + j(\gamma_1 R_2 + \gamma_2 R_1). \tag{III.13}
\]

Since our problem data are real we can equate real and imaginary parts to obtain $\Pi_1$ and $\Pi_2$ in terms of the parameters $\gamma_1$ and $\gamma_2$ as

\[
\Pi_1 = (\gamma_1 R_1 - \gamma_2 R_2), \tag{III.14} \\
\Pi_2 = (\gamma_1 R_2 + \gamma_2 R_1). \tag{III.15}
\]

We also note that if we apply $C$ to (III.12) and use (III.11) then we obtain a linear equation for $\gamma_1$ and $\gamma_2$

\[
1 = (\gamma_1 + j\gamma_2)g(j\alpha) \tag{III.16}
\]

where, as above, $\Re$ and $\Im$ denote the real and imaginary parts, and $g(s) = C(sI - A)^{-1}b, \quad s \in \mathbb{C}\setminus\sigma(A)$

In particular, based on purely elementary methods, we can compute, via a Green’s function for a second-order boundary value problem, that for $s \in \mathbb{C}\setminus\sigma(A)$,

\[
g(s) = \frac{2\sinh(\sqrt{s}/2)}{s\cosh(\sqrt{s}/2)}, \quad s \in \mathbb{C}\setminus\sigma(A).
\]

These equations allow us to explicitly compute $\gamma_1, \gamma_2, \Pi_1,$ and $\Pi_2$. Namely, we have

\[
\gamma_1 = \frac{R_1}{g(j\alpha)^2} \quad \gamma_2 = -\frac{\Im(g(j\alpha))}{\Re(g(j\alpha))^2} \tag{III.17}
\]

and

\[
\Pi_1 = \frac{R_1}{g(j\alpha)^2} R_1 - \frac{\Im(g(j\alpha))}{g(j\alpha)^2} R_2 \tag{III.18}
\]

For our specific numerical example (i.e., with $x_0 = 3/4, x_1 = 1/4, \nu_0 = \nu_1 = 1/4$) the transfer function is given by

\[
g(s) = \frac{2\sinh(\sqrt{s}/2)}{s\cosh(\sqrt{s}/2)}, \quad s \in \mathbb{C}\setminus\sigma(A).
\]

For our example we have

\[
\gamma_1 \approx -0.33187, \quad \gamma_2 \approx 2.01104.
\]

and Figs. 5 and 6 contain plots of $\Pi_1$ and $\Pi_2$.

This example provides a simple illustration of Corollary (V.2) of Section V, in which it is shown that the regulator equations are solvable if and only if the eigenvalues of the exosystem are not transmission zeros of the system. For this example, since $\alpha \neq 0$ the conditions of Corollary (V.2) are satisfied and, indeed, all the expressions make sense.

The regulator equations (III.8)–(III.10) can also be solved off-line numerically. Indeed, using numerically approximate solutions for the feedback control the resulting outputs and error
for the system with control \( u = Kz + Lw \) are depicted in Figs. 7 and 8. In Fig. 9 we have plotted the solution surface \( z(x, t) \).

The proof of our main result (Theorem IV.1) shows that the error should decay at a rate proportional to \( e^{-\beta t} \). In Fig. 7 we have plotted both \( e(t) \) and \( e^{-\beta t} \) and it is clear that these values are in line with that predicted by our main result.

We conclude that for this particular example the general method described in a detailed manner below allows us to solve the state output regulation problem and the output converges (exponentially) to the reference signal as \( t \to \infty \).

IV. THE REGULATOR EQUATIONS AND MAIN RESULTS

The main results of this section are contained in Theorems IV.1 and IV.2 which give necessary and sufficient conditions for the solvability of the state feedback and error feedback regulator problems, respectively.

**Theorem IV.1:** Let H1 and H2 hold. The linear state feedback regulator problem is solvable if and only if there exist mappings \( \Pi \) and \( \Gamma \) satisfying the “regulator equations”

\[
\Pi S = A\Pi + B\Gamma + P \\
\Pi Q = 0
\]

In this case a feedback law solving the state feedback regulator problem is given by

\[ u(t) = Kz(t) + (\Gamma - K\Pi)w(t) \]

where \( K \) is any exponentially stabilizing feedback for \( (A, B) \).

**Proof:** We first prove necessity. Assume that \( u(t) = Kz(t) + Lw(t) \) solves the linear regulator problem, i.e., 1a and 1b of Problem II.1 hold (the state feedback regulator problem is solvable).

The composite operator on \( H = \mathbb{C} \times \mathbb{H} \)

\[ A = \begin{bmatrix} A_K & (BL + P) \\ 0 & S \end{bmatrix} \]

where \( A_K = (A + BK) \), satisfies the spectrum decomposition condition for some \( \delta < 0 \), as defined in [9, pp. 71, 232]. Thus we can conclude that \( H \) decomposes into the direct sum

\[ H = H^+ \oplus H^- \]

where \( H^\pm \) are invariant subspaces under the corresponding \( C_0 \)-semigroup \( T_A(t) \) and also under \( (sI - A)^{-1} \) for \( s \in \rho(A) \). Also \( H^+ \subset D(A), A^+ \subset H^+ \), \( A(D(A) \cap H^-) \subset H^- \), and \( \dim H^+ = \dim H^- \). A restricted to \( H^+ \) has all its eigenvalues in \( \mathbb{R} \) (i.e., they coincide with the eigenvalues of \( S \)), while \( T_A \) restricted to \( H^- \) is exponentially stable.

Therefore we can define a linear operator \( \Pi \in L(H, Z) \) by the condition

\[ \left\{ \begin{bmatrix} Iw \\ w \end{bmatrix} : w \in W \right\} = H^+ \]

and we note that \( \text{Ran}(\Pi) \subset D(A) \). From the structure of \( A \) it is easy to see that

\[ \left\{ z : z \in Z \right\} = H^- . \]
For every $u_0 \in W$, from the $A$ invariance of $\mathcal{X}^+$ we can write
\[
\begin{bmatrix}
A_K \\
0
\end{bmatrix} \begin{bmatrix}
(P + BL) \\
S
\end{bmatrix} \begin{bmatrix}
\Pi u_0 \\
u_0
\end{bmatrix} = \begin{bmatrix}
\Pi S u_0 \\
Su_0
\end{bmatrix}.
\]
This implies
\[
\Pi S = A\Pi + B(L + K\Pi) + P
\]
and therefore the first regulator equation (IV.1) holds with $\Gamma = (L + K\Pi)$.

Now from the $T_A(t)$ invariance of $\mathcal{X}^+$, we can write
\[
\begin{bmatrix}
z(t) \\
u(t)
\end{bmatrix} = T_A(t) \begin{bmatrix}
\Pi u_0 \\
u_0
\end{bmatrix} = \begin{bmatrix}
\Pi \Pi T_S(t)u_0 \\
T_S(t)u_0
\end{bmatrix}
\]
where $T_S(t) = e^{St}$. Recall that $\text{Ran}(\Pi) \subset \mathcal{D}(A)$ and $C$ is continuous so that we can apply $C$ to $\Pi T_S(t)u_0$. Thus applying $[C - Q(t)]$ to (IV.4) we obtain
\[
(\Pi \Pi - Q)T_S(t)u_0 = \epsilon(t) \to 0,
\]
for every initial condition $u_0 \in W$. Since the exosystem is neutrally stable, all trajectories are bounded and almost periodic. In particular, the resulting set of $\omega$-limit points (of all trajectories) is dense in $W$ so that $\Pi \Pi - Q = 0$ on a dense subset of, and hence on all of, $W$. Therefore the second regulator equation is satisfied.

We now turn to the proof of sufficiency. As above $e^{St}$ is Lyapunov stable. Thus we suppose that $\Pi \in \mathcal{L}(W, Z)$, $\text{Ran}(\Pi) \subset \mathcal{D}(A)$, and $\Gamma \in \mathcal{L}(W, U)$ solve the regulator equations. Due to our assumption that $(A, B)$ is stabilizable, if the generator $A \Pi$ is not exponentially stable we can define $L \equiv (\Gamma - K \Pi)$ and replace the first regulator equation with
\[
(A + BK)\Pi - \Pi S = -(BL + P)
\]
where $K$ is a stabilizing feedback for $(A, B)$ so that $\sigma(S)$ is separated from $\sigma(A + BK)$. For all $w \in W$, we can write the solution $\Pi w$ to (IV.5), as
\[
\Pi w = \int_0^\infty T_K(t)(BL + P)e^{-St}w \, dt,
\]
where $T_K(t)$ is the exponentially stable semigroup generated by $(A + BK)$. First note that for every $w \in W$, the exponential stability of $T_K(t)$ and the boundedness of $e^{St}$ imply that the integrand in (IV.6) is in $L^1((0, \infty), Z)$ so the integral makes sense. Furthermore, for any $z^* \in \mathcal{D}(A^*_K)$ we have
\[
\langle \Pi w, A^*_K z^* \rangle_Z = \left\langle \int_0^\infty T_K(t)(BL + P)e^{-St}w \, dt, A^*_K z^* \right\rangle_Z
\]
\[
= \left\langle \int_0^\infty e^{-St}w, T_K(t)(BL + P)e^{St}z^* \right\rangle_Z dt
\]
\[
= \int_0^\infty \left\langle e^{-St}w, [(BL + P)*T_K(t)(BL + P)e^{St}z^*] \right\rangle_Z dt
\]
\[
= \int_0^\infty \left\langle e^{-St}w, \frac{d}{dt}[(BL + P)*T_K(t)(BL + P)e^{St}z^*] \right\rangle_Z dt
\]
\[
= \left\langle (\Pi w, (BL + P)e^{-St}w), z^* \right\rangle_Z
\]
\[
= \left\langle \Pi S w - (BL + P)w, z^* \right\rangle_Z.
\]
Hence the linear densely defined functional
\[
z \mapsto \langle \Pi w, A^*_K(z^*) \rangle_Z
\]
extends to an everywhere defined continuous linear functional and by the definition of an adjoint operator
1) $\text{Ran}(\Pi) \subset \mathcal{D}(A_K) = \mathcal{D}(A)$
2) $\langle z, [A_K\Pi - \Pi S + (BL + P)w]z \rangle_Z = 0$ for all $z \in Z$ and $w \in W$, i.e., (IV.5) holds, or equivalently (IV.1) holds with $\Gamma = K\Pi + L$.

It remains to show that with $\Gamma = K\Pi + L$ and $\Pi$, given above, satisfying the second regulator equation (IV.2), the error $\epsilon(t)$ tends to zero when $t$ tends to infinity for every initial data $z_0 \in Z$ and $u_0 \in W$. Due to the upper triangular structure of the closed-loop system we have $w(t) = e^{St}$ and applying the variation of parameter formula we have
\[
\epsilon(t) = Cz(t) - Qu(t)
\]
\[
= CT_K(t)z_0 + C\int_0^t T_K(\tau)(BL + P)w \, \tau - Qe^{St}u_0
\]
\[
= CT_K(t)z_0 + (C\Pi - Q)e^{St}u_0
\]
\[
- C\int_t^\infty T_K(\tau)(BL + P)e^{S(t-\tau)}w_0 \, \tau.
\]
The term $CT_K(t)z_0$ tends to zero as $t$ tends to infinity since $C$ is continuous and $T_K(t)$ is exponentially stable. Since, due to our assumptions, the integrand
\[
T_K(\tau)(BL + P)e^{S(t-\tau)}w_0
\]
in the last term is in $L^1((0, \infty), Z)$ and thus the last term above tends to zero as $t$ tends to infinity. Thus we conclude that
\[
\lim_{t \to \infty} \epsilon(t) = 0, \quad z_0 \in Z, \quad u_0 \in W.
\]

As an immediate consequence of the formula (IV.7) we can give bounds for the rate of decay of $\|\epsilon(t)\|_Y$.

**Corollary IV.1:** There is a positive constant $\alpha$ [which can be readily computed from (IV.7)] depending only on $\|C\|, \|z_0\|_Z, \|u_0\|_W$, so that
\[
\|\epsilon(t)\|_Y \leq \alpha e^{-\beta t}, \quad t \geq 0
\]
where $\|T_K(t)\| \leq Me^{-\beta t}$ and $\beta > 0$ since $T_K(t)$ is exponentially stable.

We now turn to the error feedback problem.

**Theorem IV.2:** Let $H_1 - H_3$ hold. The linear error feedback regulator problem is solvable if and only if there exist mappings $\Pi \in \mathcal{L}(W, Z)$ and $\Gamma \in \mathcal{L}(W, U)$ with $\text{Ran}(\Pi) \subset \mathcal{D}(A)$, such that
\[
\Pi S = A\Pi + B\Gamma + P
\]
\[
C\Pi = Q.
\]

With this $\Pi$ and $\Gamma$ a controller solving the error feedback regulator problem is given by
\[
\dot{X}(t) = FX(t) + Ge(t)
\]
\[
u(t) = HX(t)
\]
where \( X \in \mathcal{X} = \mathbb{R} \times \mathbb{R} \)

\[
G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}, \quad H = \begin{bmatrix} K & (I - KI) \end{bmatrix}
\]

(IV.12)

\[
F = \begin{bmatrix} (A + BK - G_1C) & (P + B(I - KI) + G_1Q) \\ -G_2C & (S + G_2Q) \end{bmatrix}
\]

(IV.13)

Here \( K \in \mathcal{L}(Z, U) \) is an exponentially stabilizing feedback for the pair \((A, B)\) and

\[
\begin{bmatrix} G_1 \\ G_2 \end{bmatrix}
\]

is an exponentially stabilizing output injection (such \( K \) and \( G \) exist by \( H_2 \) and \( H_3 \)).

**Proof:** Suppose the error feedback problem is solvable with the controller

\[
\dot{X}(t) = FX(t) + Ge(t) \\
u(t) = HX(t).
\]

Let

\[
\Theta = \begin{bmatrix} z \\ X \end{bmatrix} \in \mathbb{R} \times \mathcal{X}
\]

and consider the composite system

\[
\dot{\Theta} = A\Theta + Bu + Pw \\
\dot{w} = Sw
\]

where we introduced the notation

\[
A = \begin{bmatrix} A & 0 \\ GC & F \end{bmatrix}, \quad B = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad P = \begin{bmatrix} P \\ -GQ \end{bmatrix}
\]

For this system the state feedback

\[
u = K\Theta + Lw \equiv [0 \quad H]\Theta
\]

solves a regulator problem as described in Theorem IV.1 since, by our assumption

\[
e(t) \equiv CT_0\Theta_0 - Qw = [C -Q] \begin{bmatrix} z \\ X \end{bmatrix} \xrightarrow{t \to \infty} 0
\]

for every initial condition \( \Theta_0 = [x_0 \quad X_0]^T \in \mathbb{R} \times \mathcal{X} \) and \( u_0 \in W \).

Thus we can apply Theorem IV.1 to obtain the existence of a mapping

\[
\hat{\Pi} = \begin{bmatrix} \Pi \\ A \end{bmatrix} : W \to \mathbb{R} \times \mathcal{X}
\]

so that with \( L = HA \), the following regulator equations are satisfied:

\[
\hat{\Pi}w = Aw + Blw + Pw \\
u = (C\hat{\Pi} - Q)w, \quad \text{for all } w \in W.
\]

(IV.14)

(IV.15)

Equation (IV.15) is the same as

\[
C\hat{\Pi} - Q = 0
\]

which is the desired second regulator (IV.10) and the first component of (IV.14) is

\[
\Pi w = A\Pi w + BHA + P
\]

which on defining \( \Gamma = HA \) is exactly (IV.9) and we have established the necessity.

On the other hand assume that \( \Pi \) and \( \Gamma \) solve (IV.9), (IV.10) with \( \text{Ran}(\Pi) \subset (\mathcal{D}(A)) \).

Let \( \mathcal{X} = \mathbb{R} \times \mathbb{R}, X \in \mathcal{X} \), and take \( G, H, \) and \( F \) from (IV.12), (IV.13) where \( K \in \mathcal{L}(Z, U) \) is an exponentially stabilizing feedback for the pair \((A, B)\) and \( G \) is an exponentially stabilizing output injection for the pair

\[
\left( \begin{bmatrix} A & P \\ 0 & S \end{bmatrix}, \begin{bmatrix} C & -Q \end{bmatrix} \right)
\]

so that

\[
A_1 = \begin{bmatrix} (A - G_1C) & (P + G_1Q) \\ -G_2C & (S + G_2Q) \end{bmatrix}
\]

is the generator of an exponentially stable semigroup \( T_1(t) \).

Let \( X = \begin{bmatrix} z \\ \psi \end{bmatrix} \in \mathbb{R} \times \mathcal{X} \) and consider the system

\[
\frac{d}{dt} X = FX + Ge \\
u = HX
\]

where

\[
F = \begin{bmatrix} A + BK - G_1C & (P + B(I - KI) + G_1Q) \\ -G_2C & (S + G_2Q) \end{bmatrix}
\]

and

\[
H = \begin{bmatrix} K & (I - KI) \end{bmatrix}.
\]

It is convenient to introduce two auxiliary variables, \( e_z \) and \( e_w \) defined by

\[
e_z = z - \tilde{z}, \quad e_w = w - \tilde{w}, \quad \Psi = \begin{bmatrix} z \\ e_w \end{bmatrix}
\]

Let us also define \( A_K = (A + BK) \), \( C = \begin{bmatrix} C & 0 \end{bmatrix} \)

\[
A = \begin{bmatrix} A_K & 0 \\ 0 & A_1 \end{bmatrix}, \quad B = \begin{bmatrix} B \\ 0 \end{bmatrix}
\]

\[
P = \begin{bmatrix} P + B(I - KI) \\ 0 \end{bmatrix}
\]

Consider the system

\[
\dot{\Psi} = A\Psi + Bu + Pw \\
\dot{w} = Sw \\
e = C\Psi + Qw
\]

(IV.16)

We claim that the mappings

\[
\hat{\Pi} = \begin{bmatrix} \Pi \\ 0 \end{bmatrix}, \quad \hat{\Gamma} = 0
\]
satisfy the regulator equations for (IV.16), i.e.,
\[
\hat{\Pi}w = \mathcal{A}\Pi w + \mathcal{B}\Gamma w + \mathcal{P}w
\]
\[
0 = \mathcal{C}\Pi w - Qw,
\]
The second and third components in the above equation are all zero and hence the equation is satisfied in these components. As for the first component we have
\[
\Pi Sw = A\Pi w + (P + B(\Gamma - K\Pi))w
\]
or
\[
\Pi Sw = A\Pi w + Pw + B\Gamma w
\]
which is the first regulator equation for the original system. The second regulator equation is
\[
\mathcal{C}\Pi w - Qw = \mathcal{C}\Pi w - Qw = 0
\]
and so (IV.16) satisfies the regulator equations for Theorem IV.1. Thus we can appeal to Theorem IV.1 to conclude that the state feedback
\[
u = Hx = H\begin{bmatrix} z \\ \dot{z} \end{bmatrix}
\]
\[
= Kz - Ke_z - (\Gamma - K\Pi)e_w + (\Gamma - K\Pi)w
\]
solves the regulator problem for (IV.16). Thus for any initial data
\[
\Psi(0) = \begin{bmatrix} z_0 \\ * \\ * \end{bmatrix}, \quad z_0 \in \mathbb{Z}, \quad w_0 \in \mathbb{W}
\]
where the notation * represents terms whose exact value is irrelevant, we have
\[
Cz - Qw = e(t) = C\Psi - Qw \rightarrow 0 \quad t \rightarrow \infty.
\]
Notice that in both theorems the proofs were constructive in the sense that they give explicit expressions for the control input in terms of solutions to the regulator equations (IV.1), (IV.2).

V. TRANSMISSION ZEROS AND SOLVABILITY CRITERIA FOR THE REGULATOR EQUATIONS

As we have seen, the regulator equations are a system of Sylvester-type operator equations. For the example considered in Section III, these operator equations may be interpreted as a coupled system of two point boundary value problems subject to extra constraints. For this reason, it would be especially important to derive solvability criteria for the regulator equations, which would for example ensure the nonexistence of conjugate points. The fact that the solvability of the regulator problem may be expressed as a nonresonance condition between the system transmission zeros and the natural frequencies of the exosystem is well known for finite dimensional systems. In this section, we develop the nonresonance conditions for the class of distributed parameter systems discussed in Section IV.

In this section we impose the following assumption.

Assumption VI.1: For the finite-dimensional Hilbert input space \(U\) and output space \(Y\) we have
\[
\dim(U) = \dim(Y) = m.
\]

We first recall that for SISO systems transmission zeroes are defined as the zeroes of the transfer function. In the MIMO case the transfer function is an \(m \times m\) matrix given by
\[
G(s) = C(sI - A)^{-1}B.
\]
We shall assume \(\det G(s) \neq 0\). In this case, we make the following definition.

Definition VI.1: \(s_0 \in \mathbb{C}\) is a transmission zero of (II.1) if \(\det (G(s_0)) = 0\).

It is also useful to introduce the concept of an invariant zero.

Definition VI.2: \(s_0 \in \mathbb{C}\) is an invariant zero of (II.1) if the system
\[
\begin{bmatrix}
(A - s_0 I) & B \\
C & 0
\end{bmatrix}
\begin{bmatrix}
z_0 \\
u_0
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]
has a solution
\[
\begin{bmatrix} z_0 \\ u_0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\]

In the SISO case it is straightforward to show that for \(s_0 \in \rho_\infty(A)\) (the connected component of the resolvent set of \(A\) containing a right half plane, see [9, p. 70]), the concepts of transmission zeros and invariant zeros coincide (see, e.g., [19]). We include a short proof of this result in the MIMO case for completeness.

Lemma VI.1: If \(s_0 \in \rho(A)\), then \(s_0\) is a transmission zero if and only if it is an invariant zero.

Proof: Let \(s_0 \in \rho_\infty(A)\) then
\[
\ker \begin{bmatrix}
(A - s_0 I) & B \\
C & 0
\end{bmatrix} \neq \{0\}
\]
if and only if there exists
\[
\begin{bmatrix} z_0 \\ u_0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]
such that
\[
\begin{bmatrix}
(A - s_0 I) & B \\
C & 0
\end{bmatrix}
\begin{bmatrix} z_0 \\ u_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]
if and only if
\[
(A - s_0 I)z_0 = -Bu_0
\]
\[
Cz_0 = 0
\]
if and only if
\[
\begin{align*}
z_0 &= (s_0 I - A)^{-1}Bu_0, \\
Cz_0 &= 0
\end{align*}
\]
if and only if \(G(s_0)u_0 = 0\) which holds if and only if \(\det(G(s_0)) = 0\).

Lemma VI.2: If \(s_0 \in \rho(A)\) and \(s_0 \in \rho(A + BK)\) then \(s_0\) is a transmission zero of \(G\) if and only if it is a transmission zero of \(G_K\) where \(G_K(s) = C(sI - A_K)^{-1}B\).

In the classical automatic control of lumped SISO systems, it is well known that the regulator problem is solvable provided no
eigenvalue of $S$ is a transmission zero, i.e., $\lambda \in \sigma(S)$ implies $G(\lambda) \neq 0$. For distributed parameter systems, it is not immediate that $G$ would be defined at $\lambda$. Since we assume that $A$ is the infinitesimal generator of a $C_0$ semigroup and $B$ and $C$ are bounded, it is known that the transfer function $G$ exists and is defined on $\rho_0(A)$, where $\rho_0(A)$ is the connected component of the resolvent which contains infinity and intersects the positive real axis, cf. [9]. Even for a fixed system, we are of course interested in solving output regulation problems for a variety of exosystems, so that we should regard $\lambda$ as being an arbitrary point in the closed right-half plane. This observation is the basis for our first nonresonance result.

Remark V.1: We note that it follows immediately from Theorems IV.1 and IV.2 that under the hypotheses H1–H3 the state feedback regulator problem is solvable if and only if the error feedback regulator problem is solvable. Thus in providing necessary and sufficient conditions for solvability of these problems we need not distinguish between the two cases. For this reason, from now on we will only refer to the state feedback case. We are now in a position to state our first main result of Section V.

Theorem V.1: For (II.1) with exosystem (II.1)–(II.4) satisfying hypotheses H1 and H2 and the assumption that $\sigma(S) \subset \rho_0(A)$, the regulator equations (IV.1) and (IV.2) are solvable, and the output regulation via state-feedback is achievable, provided no eigenvalue of $S$ is a transmission zero, i.e., $\lambda \in \sigma(S)$ implies $\det(G(\lambda)) \neq 0$.

Proof: Suppose the regulator equations are solvable for (II.1) with exosystem (II.1)–(II.4) with feedback $u = Kz + Lw$. Consider the closed-loop system

$$\dot{z} = A_K z + Pw + BLw$$

$$e = Cz - Qw$$

(V.3)

where the “inputs” $w$ are generated by the exosystem

$$\dot{w} = Sw.$$  

We note that the transfer function from $w$ to $e$ is defined on a neighborhood of $C^+$ and is given by

$$\tilde{G}(s) = C(sI - A_K)^{-1}(P + BL) - Q.$$  

(V.5)

To say the state feedback regulator problem is solvable is to say that the steady-state response of (V.3), (V.4) to the signal $e^{St}w_0$ is zero. In particular if $w_i$ is an eigenvector of $S$ corresponding to the eigenvalue $j\alpha_i \in \mathbb{R}$, then we must have

$$G(j\alpha_i)w_i = 0, \quad i = 1, \ldots, k.$$  

(V.6)

Conversely, if (V.6) holds, then $K$ and $L$ give a solution to the output regulation problem (by state feedback).

Suppose then that $G(s)$ has no transmission zeros in the spectrum of $S$. By Lemma V.2 $G_K(s)$ has no transmission zeros in the spectrum of $S$. Therefore $G_K(j\alpha_i)$ is invertible. Now, since (V.6) is equivalent to

$$0 = G_K(j\alpha_i)Lw_i + C(j\alpha_i I - A_K)^{-1}Pw_i - Qw_i$$

$$i = 1, \ldots, k$$

(V.7)

Corollary V.1: Under the same hypotheses as the theorem, the regulator equations (IV.1) and (IV.2) are solvable, and the output regulation via state-feedback is achievable, for every choice of $P$ and $Q$ if and only if $\det(G(j\alpha)) \neq 0$ for $i = 1, \ldots, k$.

We next relax the condition relating the spectrum of the exogenous system with the component of the resolvent which contains infinity.

Definition V.3: An operator $A$ is said to satisfy the spectrum decomposition assumption with respect to the closed right half plane if $\sigma(A) = \sigma_d(A) \cup \sigma_s(A)$ where $\sigma_u(A) \cap \sigma_s(A) = \emptyset$, $\sigma_u(A) \subset \mathbb{C}^+$ consists of finitely many eigenvalues of finite multiplicity and $\sigma_s(A) \subset \mathbb{C}^{-}$.

Corollary V.2: Assume that $A$ satisfies the spectrum decomposition assumption with respect to the closed right half plane and that (II.1) with exosystem (II.1)–(II.4) satisfies hypotheses H1 and H2 of the basic Assumption II.1. The regulator equations (IV.9) and (IV.10) are then solvable, and the output regulation via state-feedback is achievable, for every choice of $P$ and $Q$ if and only if no eigenvalue of $S$ is a transmission zero, i.e., $\lambda \in \sigma(S)$ implies $\det(G(\lambda)) = C(\lambda I - A)\delta^2 - B \neq 0$.

Proof: We first note that $G$ is defined on $C^+\setminus \sigma_u(A)$, so that $G(\lambda)$ is defined for all $\lambda \in \sigma(S)$. By Lemma V.2, $\det(G(\lambda)) = 0$, for $\lambda \in \sigma(S)$ if and only if $\lambda$ is a transmission zero of $G_K$ where $K$ is a stabilizing feedback law. By Corollary V.1, the regulator equations are solvable for $(A + BK), B,C)$, $S$ and all choices of $P$ and $Q$. We also note that the solvability of the regulator equations for $(A + BK), B, C)$ implies and is implied by the solvability of the regulator equations for $(A, B, C)$. Therefore, the regulator equations are solvable if, and only if, no eigenvalue of $S$ coincides with a transmission zero of $G$.

As an example, consider the system discussed in Section III. For our specific numerical example the transfer function is given by

$$G(s) = \frac{2\sinh(\sqrt{s}/2)}{s^{1/2}\sqrt{s} \cos(\sqrt{s}/2)}.$$  

In this example we considered the problem of tracking a periodic output with frequency $\alpha$ in which case the spectrum of $S$ consists of the pair of complex numbers $\pm j\alpha$ which are assumed to be nonzero. The spectrum of $A$ consists of $\{0\} \cup \{-k^2\pi^2\}_{k \in \mathbb{N}}$. So there is one unstable eigenvalue located on the imaginary axis. According to Corollary V.2 the regulator equations are solvable if and only if

$$G(j\alpha) = \frac{2\sinh(\sqrt{j\alpha}/2)}{s^{1/2}\sqrt{s} \cos(\sqrt{j\alpha}/2)} \neq 0$$

which is obviously true for $\alpha \neq 0$.

Our final nonresonance result will remove the condition $\sigma(S) \subset \rho(A)$. This begins with the observation that one can
also express the basic nonresonance condition in terms of a
generalized Hautus test involving invariant zeros.

*Corollary V.3:* For the system \((II.1)\) with exosystem
\((II.1)\)–\((II.4)\) satisfying hypotheses \(H1\) and \(H2\), the regulator equations \((IV.9)\) and \((IV.10)\) are solvable for every choice of \(P\) and \(Q\) if and only if no eigenvalue of \(S\) is an invariant zero;
that is, if and only if
\[
\ker \left[ \begin{array}{cc}
A - \lambda I & B \\
0 & 0
\end{array} \right] = \{0\}
\text{ for all } \lambda \in \sigma(S).
\]

*Proof:* We first note that the invariant zeros of \((A, B, C)\)
coincide with the invariant zeros of \(((A + BK), B, C)\). By hypothesis we can choose \(K\) so that
\[
\|T_K(t)\| \leq Me^{-\beta t}, \quad \text{for some } \beta > 0
\]
and therefore
\[
\{\lambda \in \mathbb{C} : \Re(\lambda) > -\beta \} \subset \rho_\infty(A + BK).
\]
In particular \(G_K\) is defined on \(\sigma(S) \subset \mathbb{R}\). Since no \(\det G_K(j\omega) \neq 0\) for all \(j\omega \in \sigma(S)\) the regulator equations
hold for the triple \((A_K, B, C)\). Therefore the regulator equations hold for \((A, B, C)\), for all choices of \(P\) and \(Q\).

Conversely, if the regulator equations are solvable for
\((A, B, C)\), then they are solvable for \((A, B, C)\), and therefore
the condition \(\det G_K(j\omega) \neq 0\) for all \(j\omega \in \sigma(S)\) and
every eigenvalue of \(S\) can be an invariant zero for \((A_K, B, C)\).
Equivalently no eigenvalue of \(S\) can be an invariant for
\((A, B, C)\).

The results presented so far give necessary and sufficient
conditions for solvability of the regulator equations for every choice
of \(P\) and \(Q\). The analysis for a particular choice of \(P\) and \(Q\) is of
course more difficult. We conclude this section by giving such
an analysis for the SISO case. In this case we need to formulate
an additional resonance condition for the plant and exosystem,
which is a consequence of hypothesis \(H3\).
Rewriting the Eq. \((V.5)\) in the SISO case (where in order to draw attention to the
fact that we are in the SISO case we now denote the transfer
functions using lowercase letters)
\[
0 = g(j\omega_i)w_i
= g_K(j\omega_i)Lw_i + c(j\omega_i I - A)^{-1}Pw_i - Qw_i
\]
for \(i = 1, \ldots, k\), where \(Pw_i, Qw_i\) are known scalars and the
scalars \(Lw_i\) are to be found. As before if no eigenvalue of \(S\)
is a transmission zero of \(G_K\) (or invariant zero of \((A, B, C)\))
then these equations can be solved. Conversely if one can show
that the right-hand side of each equation is always nonzero these
sufficient conditions become necessary. Now assume hypothesis \(H3\)
concerning exponential detectability of the pair \((A, C)\) where
\[
A = \begin{bmatrix} A & P \\ 0 & S \end{bmatrix}, \quad C = \begin{bmatrix} C & -Q \end{bmatrix}. \quad (V.8)
\]
If \(\lambda \in \sigma(S)\) then \(\lambda\) is in the point spectrum of \(A\) and has an
eigenvector of the form
\[
v = \begin{bmatrix} (\lambda I - A)^{-1}Pw \\ w \end{bmatrix} \quad (V.9)
\]
where \(w\) is an eigenvector for the eigenvalue \(\lambda\) of \(S\). To say
\(Cv \neq 0\) is to say that the right-hand side of the corresponding
equation \((V.8)\) is not zero. Suppose then that \(Cv = 0\) then
\[
CT_A(t)v = e^{\lambda t}Cv = 0.
\]
Since the pair \((A, C)\) is exponentially detectable there exists a
\(G \in \mathcal{L}(Y, Z \times W)\), with
\[
G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}, \quad G_1 \in \mathcal{L}(Y, Z), \quad G_2 \in \mathcal{L}(Y, W)
\]
such that the system
\[
\dot{\Theta} = (A - GC)\Theta \quad (V.10)
\]
is exponentially stable. Since \(CT_A(t)v \equiv 0\) we see that \(\Theta(t) =
T_A(t)v\) is also a solution of \((V.10)\) and therefore it tends to zero
exponentially as \(t \to \infty\). Thus we have that \(T_A(t)v \to 0\) as \(t \to \infty\).
But, as we have already seen in Section IV, the triangular
form of \(A\) implies that \(T_A(t)\) has the form
\[
T_A(t) = \begin{bmatrix} T_A(t) & * \\ 0 & T_S(t) \end{bmatrix}
\]
and from the special form of \(v\) [in \((V.9)\)] we see that this would
imply that
\[
T_S(t)w \to 0, \quad t \to \infty
\]
which is a contradiction to our hypothesis \(H1\) which is unten-
able since \(v\) is an eigenvector for \(S\). In particular for systems
satisfying \(H1\) and \(H3\) \(Cv \neq 0\) and, therefore each right-hand
side of \((V.8)\) is nonzero.

*Corollary V.4:* Suppose the plant and exosystem satisfies
hypotheses \(H1\)–\(H3\). The regulator equations are solvable, and
output regulation by error feedback can be achieved, if and
only if no natural frequency of the exosystem is a transmission
zero of the plant, i.e., \(g(\lambda) \neq 0\) for all \(\lambda \in \sigma(S)\).

*Remark V.2:* This corollary imposes additional restrictions
on \(P\) and \(Q\), viz., hypothesis \(H3\) in order to obtain necessity of
the resonance condition and to be able to design error feedback
control schemes.

VI. EXAMPLES OF OUTPUT REGULATION

*Example VI.1—Set Point Control with Periodic Disturbance for a One-Dimensional Heat Equation:* Consider the controlled one-dimensional heat equation analyzed in Section III
with an additional external disturbance \(D(t)\)
\[
\frac{d}{dt}z(t) = Az(t) + Bu(t) + D(t) \quad (VI.1)
\]
\[
y(t) = Cz(t)
\]
\[
z(0) = \varphi.
\]
Here $A = \frac{d^2}{dx^2}$ in $Z = L^2(0, 1)$ is a self-adjoint operator with the domain,

$$\mathcal{D}(A) = \{ \varphi \in H^2(0, 1); \varphi'(0) = \varphi'(1) = 0 \}.$$ 

In this example, we consider the same one-dimensional bounded input and output operators $B$ and $C$ as in (III.4) and (III.5), respectively, so that $Y = U \subseteq \mathbb{R}$. We also use the same stabilizing feedback (III.6) given by

$$K\varphi = -\beta \langle \varphi, 1 \rangle, \quad \beta > 0$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2(0, 1)$.

For this example we are interested in controlling the output $y(t)$ to track a constant reference trajectory of the form $y_r = M$. In our work [1] we considered this example without the additional disturbance $D(t)$. In this present example we assume that the system is forced by a periodic external disturbance $d(t) = M_d \sin(\alpha t + \varphi)$ acting over a small spatial interval.

In this case we can construct a three-dimensional exogenous system

$$\dot{w} = Sw, \quad \dot{w}(0) = [M \quad u_2^0 \quad u_3^0]^T$$

with $w = [w_1 \quad w_2 \quad w_3]^T \in \mathbb{R}^3$

$$S = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \alpha \\ 0 & -\alpha & 0 \end{bmatrix}$$

and $[u_2^0 \quad u_3^0]^T \in \mathbb{R}^2$ chosen so that the first component, $w_2$, of

$$w_2 = \alpha w_3, \quad w_3 = -\alpha w_2, \quad w_2(0) = u_2^0, \quad w_3(0) = u_3^0$$

gives $w_2(t) = d(t) = M_d \sin(\alpha t + \varphi)$.

In this example $\sigma(S) = \{0\} \cup \{\pm \alpha i\} \subset \mathbb{C}$. In terms of our earlier notation, $Q = [1 \quad 0 \quad 0]$ so that $y_r(t) = Qw = M$. We also assume that the disturbance $D(t)$ only acts in a small neighborhood of the right end point of the interval (i.e., in a neighborhood of $x = 1$). In particular, we assume that

$$D(t) = Pd(t) = \chi_{[0,1]}(x) M_d \sin(\alpha t + \varphi), \quad 0 < \epsilon < 1$$

where

$$P = [0 \quad \chi_{[0,1]}(x) \quad 0] = [0 \quad p_2 \quad 0].$$

Note that since $s = 0$ is in the spectrum of $A$ we will first introduce a stabilizing feedback $K$ and replace $A$ by $(A + BK)$ so that the spectrum lies strictly in the left half-plane and the plant is exponentially stable. In this case, in applying Theorem IV.1, we obtain a control in the form $u = \Gamma w$ where

$$\Gamma = \begin{bmatrix} \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix} \in \mathbb{R}^3 \text{ and } \Pi = [\Pi_1 \quad \Pi_2 \quad \Pi_3] \in \mathcal{L}(\mathbb{R}^3, \mathbb{Z})$$

satisfy the regulator equations

$$\Pi S \Pi w = (A + BK) \Pi w + B \Gamma w + P w = 0 = C \Pi w - Qw.$$ (VI.2)

The second regulator equation (VI.3) reduces to

$$C \Pi_1 = 1, \quad C \Pi_2 = C \Pi_3 = 0.$$ (VI.7)

For all the examples given in this paper it is possible to give representations for the solutions of the regulator equations in terms of operators that can be given explicitly. However, in practice such representations are either not available or are of no practical value. Rather than pursue this approach, as we have done in Section III, we will present a simple procedure for obtaining approximate solutions that are easy to implement numerically.

For our explicit numerical simulation we have taken $K, B$ and $C$ just as in the example in Section III with $\beta = 0.5$, $x_0 = 3/4$, $v_0 = 1/4$, $\alpha_0 = 1/4$. We ask that the output $y(t) = Cz(t)$ approach the constant reference temperature $y_r(t) = 1/2$, i.e., we have set $M = 1/2$. For the disturbance we have taken $\alpha = 2, M_d = 2$ and $\varphi = 0$ so that $d(t) = 2 \cos(2t)$ and we allow the disturbance to influence the spatial interval $[3/4, 1]$ by setting, $\epsilon = 1/4$. Finally we have taken the initial condition $z(x) = 4x^2(3/2 - x)$.

In Figs. 10 and 11 we have plotted the output $y$ and approximate solution $\tilde{z}(x, t)$ for (VI.1) with $u \equiv 0$ and disturbance term $P d(t) = 2 \cos(2t) \chi_{[3/4, 1]}(x)$. 

![Fig. 10. Output $y$ with disturbance.](image1)

![Fig. 11. Solution surface $z(x, t)$.](image2)
The steady state depicted in the figure reflects the superposition of the dc-bias, due to the instability of the open-loop system and, of the periodic disturbance.

In Figs. 12 and 13 we add the stabilizing feedback \( u = Kz = -\beta(z, 1) \) with \( \beta = \gamma \) and, in this case, we see that dc-bias has been attenuated so that the solution approaches a period motion about the line \( z = 0 \).

In Fig. 14, we have introduced the control law \( u = \Gamma w \) (with the approximate \( \Gamma \) computed above) and we have plotted the exact reference trajectory \( y_r = \gamma \) and the numerically computed output \( y \) using the approximate feedback control law described above. Fig. 14 contains a plot of the approximate numerical solution \( z(x, t) \) for \( (x, t) \in [0, 1] \times [0, 10] \) for (VI.1). We have chosen a rather large amplitude disturbance in order to draw attention to the fact that even though we have a large disturbance over the interval \([3/4, 1]\), the output, which is the average temperature over the interval \([0, 1/2]\) converges very rapidly to the required value of \( M = \gamma \) on the interval \([0, 1/2]\).

**Example VI.2—Error Feedback Control for a One-Dimensional Heat Equation:** Consider again the controlled one-dimensional heat equation on a finite rod

\[
\frac{d}{dt} z(t) = Az(t) + Bu(t)
\]

\( y(t) = Cz(t) \)

\( z(0) = \phi \)

where as in the previous example \( A = d^2/dx^2 \) with Neumann boundary conditions and \( B \) and \( C \) are given by (III.4) and (III.5), respectively.

In this example our objective is to design a dynamic controller that will force the the output of the composite system to track a periodic reference signal \( y_r(t) = M \sin(\alpha t) \). Thus the basic setup is exactly the same as the motivating example given in Section III. Just as in the motivating example we may take the exogenous system in (II.4) to be a harmonic oscillatory

\[
\psi = Sw, \quad S = \begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix}, \quad w(0) = \begin{pmatrix} 0 \\ M \end{pmatrix}.
\]

Here, \( Q = [1, 0], P = 0 \), and \( S \in \mathcal{L}(W) \) with \( k = 2 \).

The main difference is that in this example we assume that only the error \( \epsilon(t) = y(t) - y_r(t) \) is available to design our control. In this case we will employ the results of Theorem (IV.2) for the error feedback regulation problem. Thus we seek a dynamic error feedback controller in the form

\[
\dot{X} = FX + Ge \quad \text{where } X \in L^2(0, 1) \times \mathbb{R}^2, \quad u = HX
\]

where \( G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} : \mathbb{R} \to L^2 \times \mathbb{R}^2 \).
is an exponentially stabilizing output injection for the pair
\[
\begin{bmatrix}
A & P \\
0 & S
\end{bmatrix} [C & -Q].
\]

The output operator \( H \) is given in terms of solutions to the regulator equations and a stabilizing feedback \( K \) for \((A, B)\) (which in this example we take to be the same as in the motivating example in Section III and in the previous example). In particular,
\[
H = [K & (\Gamma - KII)].
\]
The operators \( \Pi = [\Pi_1, \Pi_2] \in \mathcal{L}(\mathbb{R}^2, Z) \) and \( \Gamma = [\gamma_1, \gamma_2] \in \mathbb{R}^2 \) are the solutions of the regulator equations given in Section III (cf. (IV.9) and (IV.10) with \( P = 0 \)).

As we have already observed in Section III, the first regulator equation reduces to the coupled system of second order ordinary differential equations (III.8), (III.9) with boundary conditions (III.10).

The parameters \( \gamma_1 \) and \( \gamma_2 \) are chosen to satisfy the second regulator equation, which in this case reduces to the additional constraints (III.11).

Having computed \( \Pi \) and \( \Gamma \) we can write [see (IV.13)]
\[
F = \begin{bmatrix}
A + BK - G_1 C & B(\Gamma - KII) + G_1 Q \\
-G_2 C & S + G_2 Q
\end{bmatrix}
\]
where for this example we can choose the stabilizing output injection \( G_1 \) for \((A, C)\) as
\[
(G_1 \psi)(x) = c \psi(x), \quad \text{for } x \in [0, 1], \, \psi \in Y, \, c > 0.
\]
(Here \( \psi(x) \) is the identically one function.)

We also need to choose
\[
G_2 = \begin{bmatrix}
\frac{\partial^4}{\partial x^4} \\
\frac{\partial^2}{\partial x^2}
\end{bmatrix}
\]
so that
\[
\begin{bmatrix}
A - G_1 C & G_1 Q \\
-G_2 C & CS + G_2 Q
\end{bmatrix}
\]
is exponentially stable. Using the piecewise linear splines to approximate functions in the \( D(A) \) we find that \( g_1^2 = g_2^2 = -3 \) and \( \epsilon = 1.9 \) provide stability with a stability margin of approximately \(-1.624\).

Using this controller and introducing the observer variables \( \hat{z} \) and \( \hat{w} \), we obtain the closed-loop system
\[
\frac{d}{dt} \begin{bmatrix}
z \\
& \hat{z} \\
& \hat{w}
\end{bmatrix} = A \begin{bmatrix}
z \\
& \hat{z} \\
& \hat{w}
\end{bmatrix}
\]
with
\[
A = \begin{bmatrix}
A & BK & B(\Gamma - KII) & G_1 Q \\
G_1 C & A & B(\Gamma - KII) + G_1 Q & -G_2 Q \\
-G_2 C & 0 & S + G_2 Q & -G_2 Q \\
0 & 0 & 0 & S
\end{bmatrix}
\]
\[
A_K = A + BK.
\]

In the numerical simulation we have used a numerical approximation for \( \Gamma \) in the feedback law and set \( x_0 = 3/4, \, x_1 = 1/4 \),
In order to formulate this problem within the current framework we proceed in the usual way and define $A = \frac{d^2}{ds^2}$ in $Z = L^2(0,1)$ as the selfadjoint operator defined by

$$\mathcal{D}(A) = \{ \phi \in H^2(0,1); \phi(0) = \phi(1) = 0 \} = H^2(0,1) \cap H^1_0(0,1).$$

Now define the space $\mathcal{H} = H^2_0(0,1) \oplus L^2(0,1) = \mathcal{D}(\sqrt{-A}) \oplus L^2(0,1)$

with the following norm:

$$||\Phi||^2_{\mathcal{H}} = ||\phi_1||^2 + ||\phi_2||^2$$

for $\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \in \mathcal{H}$. Note that this norm is equivalent to the graph norm.

Next we define the operator $A$ with $\mathcal{D}(A) = \mathcal{D}(\sqrt{-A}) \oplus H^1_0$ by

$$A = \begin{pmatrix} 0 & I \\ -2\partial \end{pmatrix}.$$

Thus we can write (VI.11) in the form

$$\dot{Z} = AZ + Bu, \quad Z(0) = \begin{pmatrix} \phi \\ \psi \end{pmatrix}, \quad y = CZ \quad (VI.12)$$

where

$$Z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} z \\ \dot{z} \end{pmatrix},$$

$$Bu = \begin{pmatrix} 0 & Bu \end{pmatrix}, \quad CZ = \begin{pmatrix} C & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = Cz_1.$$

and $B$ and $C$ are given by (III.4) and (III.5).

It is straightforward to verify that $A$ is maximal dissipative and therefore generates a contraction semigroup. In fact more is true, namely, $A$ generates an exponentially stable semigroup.

This can be seen by direct estimates or one can first show that $A$ is a Riesz spectral operator satisfying the spectrum determined growth condition (see for example Curtain–Zwart [9, Th. 2.3.5.c]), the spectrum lies along the line $\Re \lambda = -\beta$ therefore the semigroup generated by $A$ is exponentially stable. Since this semigroup is already stable, in Theorem 4.1 we can choose $K = 0$. Therefore we need only compute the mappings $\Pi$ and $\Gamma$ solving the regulator equations and take the feedback law $u = \Gamma w$.

To this end we seek linear operators $\Pi = [\Pi_1 \quad \Pi_2]: \mathbb{R}^2 \rightarrow \mathcal{H}$, and $\Gamma = [\Gamma_1 \quad \Gamma_2]: \mathbb{R}^2 \rightarrow U$ satisfying

$$\Pi w = \mathcal{A}\Pi w + \mathcal{B} \Gamma w,$$

$$\mathcal{C} \Pi w - Q w = 0,$$

for $w \in \mathbb{R}^2$.

Note that since $\Pi = [\Pi_1 \quad \Pi_2]$ the second regulator equation becomes

$$0 = (\mathcal{C} \Pi - Q) w = \mathcal{C} \Pi_1 w_1 + \mathcal{C} \Pi_2 w_2 - w_2$$

which implies

$$\mathcal{C} \Pi_1 = 1, \quad \mathcal{C} \Pi_2 = 0. \quad (VI.13)$$

The first regulator equation gives

$$\alpha \Pi_1 w_2 - \alpha \Pi_2 w_1 = \mathcal{A} \Pi_1 w_1 + \mathcal{A} \Pi_2 w_2 + \mathcal{B} \Gamma_1 w_1 + \mathcal{B} \Gamma_2 w_2$$

which can be written as

$$\mathcal{A} \Pi_1 + \alpha \Pi_2 = -\mathcal{B} \Gamma_1 \quad (VI.14)$$

$$\mathcal{A} \Pi_2 - \alpha \Pi_1 = -\mathcal{B} \Gamma_2. \quad (VI.15)$$

Using this notation we can write (VI.13)–(VI.15) in the form

$$\Pi_1 = \begin{pmatrix} \Pi_1^1 \\ \Pi_1^2 \end{pmatrix}, \quad \Pi_2 = \begin{pmatrix} \Pi_2^1 \\ \Pi_2^2 \end{pmatrix}.$$

As we have already mentioned, since the original system is exponentially stable we may use a control $u = \Gamma w$. Thus the resulting closed-loop system can be written as

$$\dot{z}_1 = z_2,$$

$$\dot{z}_2 = A z_1 - 2 \beta z_2 + [\mathcal{B} \Gamma_1 \quad \mathcal{B} \Gamma_2] \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \quad (VI.16)$$

$$\dot{w}_1 = \alpha w_2, \quad (VI.17)$$

$$\dot{w}_2 = -\alpha w_1. \quad (VI.18)$$

In our numerical example we have set $z_0 = 3/4, z_1 = 1/4, \nu_0 = \nu_2 = 1/4, M = 1, \alpha = 1, \beta = 5$, and chosen initial conditions $\phi(x) = 16x^2(1-x)^2$ and $\psi = 5\sin^2(\pi x)$.

Fig. 18 depicts the reference signal $\gamma_0(t) = \sin(t)$ and the controlled output $y(t)$ for the closed-loop system (VI.16). Finally, Fig. 19 contains the numerical solution for the displacement $z_1$ for $x \in [0,1]$ and $t \in [0,15]$. 
VII. CONCLUSION

This work extends the geometric theory of output regulation to linear distributed parameter systems with bounded input and output operators, in the case when the reference signal and disturbances are generated by a finite dimensional exogenous system. It is shown that the full state feedback and error feedback regulator problems are solvable, under the standard assumptions of stabilizability and detectability, if and only if a pair of regulator equations is solvable. The regulator equations form a system of Sylvester-type operator equations subject to extra side constraints.

Concerning solvability of the regulator equations, it is well known for finite-dimensional systems that solvability of the regulator problem may be expressed as a nonresonance condition between the system transmission zeros and the natural frequencies of the exosystem. In Section V we have also developed such nonresonance conditions for the class of distributed parameter systems discussed in Section IV.

Several examples are given to demonstrate applications of the main results. Using the regulator equations to design state and error feedback control laws we solve a number of regulator problems (with and without additional disturbances) for parabolic and hyperbolic partial differential control systems. For each of these examples the regulator equations reduce to a system of linear ordinary differential equations which can, in general, be readily solved numerically off-line to obtain approximate feedback controls that work very well in practice.

In future work the authors plan to carry out a nontrivial extension of this work to the case of unbounded input and outputs operators (i.e., boundary control and point actuators and sensors) and also the case of infinite-dimensional exosystems (e.g., repetitive control).

ACKNOWLEDGMENT

The authors would like to express their gratitude to the anonymous referees for their detailed review of the original version of this manuscript and for offering many useful comments and suggestions. They would particularly like to thank P. Grabowski for his extremely detailed review and for many useful suggestions including the alternative proof of sufficiency in Theorem IV.1 used in this final version of the paper.

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