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STEADY-STATE SOLUTIONS OF MAXWELL'S EQUATIONS OVER A THREE-DIMENSIONAL CONDUCTING HALF SPACE

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Abstract

A rigorous derivation is given of steady-state solutions $v(x)$ to the problem of the electromagnetic field over a three-dimensional, conducting half space $\mathbb{R}^3_-$ due to a time-harmonic source located in $\mathbb{R}^3_+$ ($\mathbb{R}^3_\pm = \{x \in \mathbb{R}^3 : \pm x_3 > 0\}$). For $x \in \mathbb{R}^3_+$ a general solution $v(x)$ has the structure

$$v = v_0 + v_{s_0} + v_{s_+} + v_{s_-} + \overset{o}{v}_p + v_p,$$

where v_0 is the static component, v_{s_0} is the AH-wave component, $v_{s_\pm}$ are the surface-wave components, and $\overset{o}{v}_p$ and v_p are the spatial-wave components obtained as superpositions of modes having the frequencies of the upper and lower media respectively. For large $|x|$ they decay as follows: v_0, $v_{s_\pm}$, $v_p = O(|x|^{-2})$, $v_{s_0} = o(|x|^{-1})$, $\overset{o}{v}_p = O(|x|^{-1})$. An explicit expression is given for the leading term of the asymptotics of $\overset{o}{v}_p$. In particular, contrary to popular belief, there is no component which decays like $O(|x|^{-1/2})$ along the interface $\{x_3 = 0\}$.

1 Introduction

For Maxwell's equations over a conducting half space the solution $u(x,t)$ for quite general sources $f(x,t)$ can be represented by Duhamel's principle [3, p. 197]. Time-harmonic solutions arise as approximations to such solutions for large t when the source is time-harmonic.

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For such approximations it is often possible to obtain much more information than for the actual solution. For this reason it is important to give a rigorous derivation of such solutions and determine their properties.

To formulate the problem let I_3 be the 3×3 identity matrix, let $\underline{O}$ be the 3×3 zero matrix, and let B and E be the diagonal matrices

$$B = \mathrm{diag}(-i\sigma I_3, \underline{O}),$$

$$\begin{aligned} E(x) &= \mathrm{diag}[\varepsilon(x) I_3, \mu(x) I_3] \\ &= \chi_+(x_3)\,\mathrm{diag}[\varepsilon_0 I_3, \mu_0 I_3] + \chi_-(x_3)\,\mathrm{diag}[\varepsilon I_3, \mu I_3] \\ &\equiv E_+ + E_-. \end{aligned}$$

Here $\chi_\pm(x_3)$ are the characteristic functions of the half spaces $\mathbb{R}^3_\pm = \{x \in \mathbb{R}^3 : \pm x_3 > 0\}$, and ε_0, μ_0 and ε, μ, σ , respectively, are the electromagnetic constants of the media filling $\mathbb{R}^3_+$ and $\mathbb{R}^3_-$; σ is the conductivity, and we assume that $c_0^2 = (\varepsilon_0\mu_0)^{-1} > c^2 = (\varepsilon\mu)^{-1}$. Let $A(D)$, $D_j = -i\partial_j, \quad j = 1, 2, 3$, be the 6×6 matrix operator

$$A(D) = A_j D_j = \begin{bmatrix} \underline{O} & i\ \mathrm{rot} \\ -i\ \mathrm{rot} & \underline{O} \end{bmatrix}, \ \mathrm{rot} = \begin{bmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{bmatrix}. \tag{1.1}$$

Setting

$$\begin{array}{rcl} L_0(D) &=& E_+^{-1} A(D), \\ L_\sigma(D) &=& E_-^{-1}[A(D) + B], \\ \Lambda(D) &=& \chi_+(x) L_0(D) + \chi_-(x) L_\sigma(D), \end{array} \tag{1.2}$$

the basic problem is to find a function $u(x,t)$, differentiable with respect to t with values in $\mathcal{D}(\Lambda)$, the domain of Λ (see §2), for each t such that for $f \in C_0^\infty(\mathbb{R}^3_+), \nu > 0$

$$i\partial_t u(x,t) - \Lambda(D) u(x,t) = -f(x)\exp(-i\nu t). \tag{1.3}$$

If now $v(x)$ is a solution of

$$[\Lambda(D) - \nu I] v(x) = f(x), \tag{1.4}$$

then

$$\tilde{u}(x,t) = v(x)\exp(-i\nu t) \tag{1.5}$$

is a time-harmonic solution of (1.3). It is, of course, not equal to the solution $u(x,t)$, since $v(x)$ is not in $\mathcal{D}(\Lambda)$ (it doesn't even have finite energy), but for large t it is reasonable to expect that it may be an approximation to $u(x,t)$ in some sense. This is what is behind the fact that in the engineering and physics literature the problem is often taken in the form

(1.4), (1.5) forthwith. The rigorous justification for this constitutes the principle of limiting amplitude (cf. [9]).

To solve (1.4) we set $\zeta = \nu + i\epsilon$, $\epsilon \in (0, \epsilon_0]$, $\epsilon_0 > 0$, solve

$$[\Lambda(D) - \zeta I]v(x; \zeta) = f(x), \tag{1.6}$$

and pass to the limit $v(x) \equiv v(x; \nu + i0)$ (the principle of limiting absorption).

We remark that in problems (1.4), (1.6) the condition is usually imposed that the tangential components of the electric and magnetic fields be continuous across the interface $\{x_3 = 0\}$. This condition is satisfied in the case (1.6), since $v(x; \zeta)$ is in $\mathcal{D}(\Lambda)$, and anything in $\mathcal{D}(\Lambda)$ satisfies this condition automatically (see §2 below). It is satisfied for $v(x)$ of (1.4), since it is the pointwise limit of the solution of (1.6).

Anticipating somewhat the notation introduced in subsequent sections, to placate the impatient reader we immediately formulate the basic result of the work. The proof forms the content of §3 − §7. It is assumed here and below that $\mu = \mu_0$, this apparently being justified on physical grounds.

Theorem 1.1 *For $f \in C_0^\infty(\mathbb{R}^3_+, \mathbb{C}^6)$, $\nu > 0$, $x \in \mathbb{R}^3_+$ a smooth solution of (1.4), $v(x) \equiv v(x; \nu + i0)$, exists and has the following form:*

$$v(x) = v_0(x) + v_{s_0}(x) + v_{s_+}(x) + v_{s_-} + \overset{o}{v}_p(x) + v_p(x). \tag{1.7}$$

Here $v_0(x)$ is the static component of the solution, while v_{s_0} is the AH-component, consisting of a superposition of modes whose phase has no real part, so that, e.g., $v_{s_0}(x)\exp(-i\nu t)$ does not propagate. The functions $v_{s_\pm}(x)$ are the surface-wave components of the solution, while $\overset{o}{v}_p(x)$ and $v_p(x)$ are the spatial-wave components consisting of superpositions of modes with the frequencies of the upper and lower media respectively. For large $|x|$, $x \in \mathbb{R}^3_+$, the behavior of the various components is as follows:

$$\begin{aligned} v_0(x),\ v_{s_\pm}(x),\ v_p(x) &= O(|x|^{-2}), \\ v_{s_0}(x) &= o(|x|^{-1}), \end{aligned}$$

$$\begin{aligned} \overset{o}{v}_p(x) &= 2g(x; \nu)\, \overset{o}{P}_1(\omega)[\theta(\omega; \nu) - \\ &\quad C_{0,1}(\nu\tilde{\omega}/c_0)]\theta(\tilde{\omega}, \nu) + O(|x|^{-1-\kappa}), \\ g(x; \nu) &= \exp(i\nu|x|/c_0)/(4\pi|x|) \\ \overset{o}{P}_1(\omega) &\quad \text{is the projection of § 2}, \\ \theta(\omega; \nu) &= (\nu/c_0^2) \int \exp(-i\omega\nu y/c_0)\, f(y)\, dy, \end{aligned} \tag{1.8}$$

$$\omega = x/|x| = (\omega_1, \omega_2, \omega_3), \quad \tilde{\omega} = (\omega_1, \omega_2, \omega_{-3}),$$

and $C_{0,1}$ is the matrix reflection coefficient of § 6, and $\kappa \in (1/2, 1)$.

Remark 1 From the theorem it follows that the leading term is $\overset{o}{v}_p(x)$, and $\overset{o}{v}_p(x)\exp(-i\nu t)$ is an outgoing hemispherical wave in $\mathbb{R}^3_+$. We have considered the solution only for $x \in \mathbb{R}^3_+$, since it is the case of greatest physical interest.

Finally, to conclude this section we briefly discuss the origin of this problem and compare our results above with earlier results. A more complete historical sketch with extensive references can be found in [1, 12].

The present problem arose in the earliest days of radio in an attempt to understand how radio waves propagate from a transmitter to a receiver while overcoming the effect of the earth's curvature (the role of the ionosphere was unknown at that time). In 1907 J. Zenneck found that in two dimensions equation (1.4) has solutions which decay exponentially away from the interface $\{x_3 = 0\}$. In 1909 A. Sommerfeld [11] made an extensive study of the steady-state field due to a vertical dipole. Because of cylindrical symmetry, this problem reduces to a problem for the scalar Helmholz equation in two independent variables. Sommerfeld obtained a representation of the solution which, when written in a particular way, seemed to display a component which he identified with Zenneck's surface wave. In [11] Sommerfeld asserted that his eventual surface-wave component decays only like $O(|x|^{-1/2})$ along the interface $\{x_3 = 0\}$, while we assert that the actual surface-wave component decays like $O(|x|^{-2})$. What is the reason for this discrepancy? Quite aside from mathematical inconsistencies in [11], as we have pointed out previously [3, 4, 5], a formal Ansatz approach, considered appropriate to a particular source or initial data, leads to results which are Ansatz-dependent: with one Ansatz the solution may seem to contain surface waves with certain properties, while with another these properties or the surface waves themselves may not be evident (cf. [6]). Thus, Weyl [13] via another Ansatz obtained a solution which showed no evidence of Sommerfeld's purported surface wave. The actual structure of the solution of problem (1.3) remained equivocal up to the appearance of our monograph [3]. For example, although very many papers have been devoted to this problem since 1909, the AH-wave component of the solution was first discovered in [3], i.e., nearly eighty years after Sommerfeld's paper. Actually, radio engineers have known for some time, at least in special instances, that the surface waves decay faster than the spatial wave (cf. [7]). Our theorem asserts that it is always the case that they decay twice as fast at least.

Finally, we mention that the analogues of (1.4) for elastic waves in $\mathbb{R}^3_+$ with a boundary free of normal stresses [10] and for Maxwell's equations in $\mathbb{R}^3_+$ with the self-adjoint Leontovich boundary conditions [9] do admit surface waves which decay only like $O(|x|^{-1/2})$ along $\{x_3 = 0\}$. For the Rayleigh wave of elasticity theory this has long been known. The reason for the disparity with the case of the surface waves of the present problem is very simple. For example, the Rayleigh wave is formed as a superposition of modes with real frequencies: in constructing a solution of the analogue of (1.4) via (1.6) the integrand for the limit function has a singularity on R_+, since the range of the frequencies of the Rayleigh modes contains the frequency ν. Handling this singularity in the usual way and using stationary phase show that the steady-state Rayleigh wave decays only like $O(|x|^{-1/2})$ [10]. In the present case the complex-valued frequencies of the surface modes are always a positive distance from ν, and the modes for (1.6) thus have no singularity. The asserted decay can be proved simply by integration by parts (once a considerable amount is known regarding the structure of the solution). This is carried out in §4, while §3, §5 − §7 contain the remainder of the proof of the theorem. The second section contains the required background material.

2 Background

In this section we present a brief outline of how the machinery for solving (1.4) was developed in [3] and along the way introduce mathematical objects needed in subsequent sections.

The operator A of (1.1) with domain $\mathcal{D}(A) = \{f \in L_2 : Af \in L_2\}$ is selfadjoint in $L_2(\mathbb{R}^3; \mathbb{C}^6)$. Let $\overset{o}{\mathcal{K}}$ ($\mathcal{K}$) denote the Hilbert space of functions $f,\ g \in L_2(\mathbb{R}^3, \mathbb{C}^6)$ with the E_+ (E_-) inner product

$$(f, g)_{\overset{o}{\mathcal{K}}} =< f, E_+ g >,\ ((f, g)_{\mathcal{K}} =< f, E_- g >))$$

where here and henceforth $< \cdot, \cdot >$ is the usual L_2 inner product with corresponding norm $\|\cdot\|$. The operator L_0 (L_σ) of (1.2) with domain $\mathcal{D}(L_0) = \mathcal{D}(A)$ ($\mathcal{D}(L_\sigma) = \mathcal{D}(A)$) is selfadjoint (maximal dissipative).

The symbol $L_0(\eta),\ \eta \in \mathbb{R}^3\backslash\{0\}$, of L_0 is

$$L_0(\eta) = \begin{bmatrix} \underline{O} & -\varepsilon_0^{-1}\eta\wedge \\ \mu_0^{-1}\eta\wedge & \underline{O} \end{bmatrix},\ \eta\wedge = \begin{bmatrix} 0 & -\eta_3 & \eta_2 \\ \eta_3 & 0 & -\eta_1 \\ -\eta_2 & \eta_1 & 0 \end{bmatrix} \tag{2.1}$$

and has eigenvalues $\overset{o}{\lambda}_j (\eta) = jc_0|\eta|,\ j = 0, \pm 1,\ c_0^2 = (\varepsilon_0\mu_0)^{-1}$, each of multiplicity two. The corresponding projections $\overset{o}{P}_j (\omega)$,

$$L_0(\eta) \overset{o}{P}_j (\omega) =\overset{o}{\lambda}_j (\omega) \overset{o}{P}_j (\omega),\ j = 0, \pm 1,\ \omega = \eta/|\eta| \in S^2,$$

are mutually orthogonal orthoprojectors with respect to the E_+ inner product:

$${}^t(E_+ \overset{o}{P}_j) = E_+ \overset{o}{P}_j,\ \delta_{jk} \overset{o}{P}_j=\overset{o}{P}_j\overset{o}{P}_k,\ j, k = 0, \pm 1.$$

They are needed explicitly:

$$\overset{o}{P}_0 (\omega) = \begin{bmatrix} {}^t\omega\omega & 0 \\ 0 & {}^t\omega\omega \end{bmatrix}, \tag{2.2}$$

$$\overset{o}{P}_j (\omega) = \begin{bmatrix} -\omega \wedge \omega\wedge & j\mu_0 c_0\omega\wedge \\ j\varepsilon_0 c_0\omega\wedge & -\omega \wedge \omega\wedge \end{bmatrix},$$

$$j = \pm 1,\ \eta \in \mathbb{R}^3\backslash\{0\},\ \omega = \eta/|\eta| \in S^2$$

The resolution of the identity for $L_0(\eta)$ is

$$
\begin{aligned}
I &= \overset{o}{P}_0(\omega) + \overset{o}{P}_1(\omega) + \overset{o}{P}_{-1}(\omega), \\
L_0(\eta) &= \overset{o}{\lambda}_1(\eta)\, \overset{o}{P}_1(\omega) + \overset{o}{\lambda}_{-1}(\eta)\, \overset{o}{P}_{-1}(\omega).
\end{aligned} \tag{2.3}
$$

The symbol $L_\sigma(\eta)$, $\eta \in \mathbb{R}^3\backslash\{0\}$, of L_σ is

$$
L_\sigma(\eta) = E_-^{-1} \begin{bmatrix} -i\sigma I_3 & -\eta\wedge \\ \eta\wedge & \underline{O} \end{bmatrix}
$$

$$
\det[L_\sigma(\eta) - \zeta I] = \zeta(\zeta + i\sigma/\varepsilon)\,[\zeta - \lambda_1(\eta)]^2\,[\zeta - \lambda_{-1}(\eta)]^2
$$

with eigenvalues

$$
\begin{gathered}
\lambda_0 = 0,\ \lambda_\sigma = -i\sigma/\varepsilon,\ \lambda_j(\eta) = -i\sigma/2\varepsilon + j\lambda(\eta), \\
\lambda(\eta) = \sqrt{[c^2|\eta|^2 - \sigma^2/4\varepsilon^2]},\ \operatorname{Im}\lambda(\eta) \geq 0,\ j = \pm 1,
\end{gathered}
$$

and corresponding projections

$$
\begin{gathered}
P_0(\omega) = \begin{bmatrix} \underline{O} & \underline{O} \\ \underline{O} & {}^t\omega\omega \end{bmatrix}, \quad P_\sigma(\omega) = \begin{bmatrix} {}^t\omega\omega & \underline{O} \\ \underline{O} & \underline{O} \end{bmatrix}, \\
P_j(\eta) = (j/2\lambda(\eta)) \begin{bmatrix} -\lambda_j(\eta)\omega\wedge\omega\wedge & -\varepsilon^{-1}\eta\wedge \\ \mu^{-1}\eta\wedge & -[\lambda_j(\eta) + i\sigma/\varepsilon]\omega\wedge\omega\wedge \end{bmatrix}, \\
j = \pm 1,\ c|\eta| \neq \sigma/2\varepsilon,
\end{gathered}
$$

having the properties

$$
{}^t(E_- P_k) = E_- P_k, \quad \delta_{kj} P_k = P_k P_j,\ \ j,k = 0, \sigma, \pm 1.
$$

The resolution of the identity for $L_\sigma(\eta)$ is

$$
\begin{aligned}
I &= P_0(\omega) + P_\sigma(\omega) + P_1(\eta) + P_{-1}(\eta), \\
L_\sigma(\eta) &= \lambda_\sigma P_\sigma(\eta) + \lambda_1(\eta) P_1(\eta) + \lambda_{-1}(\eta) P_{-1}(\eta).
\end{aligned} \tag{2.4}
$$

We observe that $\lambda_1(\eta) = \lambda_{-1}(\eta)$ on the sphere $\{c|\eta| = \sigma/2\varepsilon\}$, and $\lambda_{\pm 1}(\eta)$, $\eta \in \mathbb{R}^3$, are thus branches of a single function. This is reflected in the singularity of $P_j(\eta)$, $j = \pm 1$, on the sphere $\{c|\eta| = \sigma/2\varepsilon\}$ where $\lambda(\eta) = 0$. Now

$$
P_1(\eta) + P_{-1}(\eta) = \begin{bmatrix} -\omega\wedge\omega\wedge & \underline{O} \\ \underline{O} & -\omega\wedge\omega\wedge \end{bmatrix}
$$

$$
\begin{aligned}
\lambda_1(\eta) P_1(\eta) + \lambda_{-1}(\eta) P_{-1}(\eta) &= (-i\sigma/2\varepsilon)\,[P_1(\eta) + P_{-1}(\eta)] \\
&\quad + \lambda(\eta)\,[P_1(\eta) - P_{-1}(\eta)]
\end{aligned}
$$

have no singularity, so that the last two terms in the resolution above are to be read together. Likewise, in the resolution of the identity for the operator L_σ in $\mathcal{K}$,

$$f = \Phi^* \left[P_0 + P_\sigma + P_1 + P_{-1}\right] \Phi f, \; f \in \mathcal{K},$$

$P_1 + P_{-1}$ is to be treated as a single projection. Here and below Φ denotes the three-dimensional Fourier transform: for $n > 1$

$$\Phi_n f(\eta) = (2\pi)^{-n/2} \int_{R^n} \exp(-i\eta x)\, f(x)\, dx$$

with inverse $\Phi^* f(p) = \Phi f(-p)$. For $n = 3$ we write simply Φ in place of Φ_3.

The spectra of L_0, L are

$$\begin{aligned} \sigma(L_0) &= \mathbb{R}, \\ \sigma(L_\sigma) &= \{-i\sigma/2\varepsilon + \lambda,\ \lambda \in \mathbb{R}\} \cup \{-i\lambda :\ \lambda \in [0, \sigma/\varepsilon]\}, \end{aligned} \tag{2.5}$$

whereby 0 is an imbedded eigenvalue for L_0, and 0 and $-i\sigma/\varepsilon$ are imbedded eigenvalues for L_σ. For $\widetilde{\zeta} \notin \sigma(L_0)$, $\zeta \notin \sigma(L_\sigma)$ the resolvents $I_0(\widetilde{\zeta}) = \left[L_0 - \widetilde{\zeta} I\right]^{-1}$, $I(\zeta) = [L_\sigma - \zeta I]^{-1}$ can be expressed in terms of their symbols by

$$\begin{aligned} I_0(\widetilde{\zeta}) f &= \Phi^* \left[L_0(\cdot) - \widetilde{\zeta} I\right]^{-1} \Phi f \equiv \int I_0(x, y; \widetilde{\zeta})\, f(y)\, dy, \\ I(\zeta) f &= \Phi^* \left[L_\sigma(\cdot) - \zeta I\right]^{-1} \Phi f \equiv \int I(x, y; \zeta)\, f(y)\, dy. \end{aligned} \tag{2.6}$$

We note that in

$$[L_\sigma(\eta) - \zeta I]^{-1} = -\zeta^{-1} P_0(\eta) + (\zeta + i\sigma/\varepsilon)^{-1} P_\sigma(\eta) +$$
$$[\lambda_1(\eta) - \zeta]^{-1} P_1(\eta) + [\lambda_{-1}(\eta) - \zeta]^{-1} P_{-1}(\eta)$$

the last two terms are to be read as

$$\frac{[-(\zeta + i\sigma/2\varepsilon)[P_1(\eta) + P_{-1}(\eta)] + \lambda(\eta)[P_{-1}(\eta) - P_1(\eta)]}{[\lambda_1(\eta) - \zeta][\lambda_{-1}(\eta) - \zeta]}$$

in a neighborhood of the sphere $\{c|\eta| = \sigma/2\varepsilon\}$, so there is no singularity there. The adjoint $L_\sigma^* = E_-^{-1}[A - B]$ of L_σ has symbol $L_\sigma^*(\eta) = \overline{L_\sigma}(\eta)$ with eigenvalues and corresponding projections which are the complex conjugates of those for L_σ, and its spectrum $\sigma(L_\sigma^*) = \overline{\sigma(L_\sigma)}$. For $\zeta \notin \sigma(L_\sigma)$ the resolvent kernel for L_σ^* is

$$I'(\cdot, \overline{\zeta}) = (2\pi)^{-3/2} \Phi^* [\overline{L_\sigma} - \overline{\zeta} I]^{-1}.$$

The importance of these resolvents in the present problem (as in others) is that they play the role of "incident" waves for the construction of the resolvent of interest in solving problem

(1.6). Namely, from the representations (2.6) it is possible to compute a very simple expression for the trace of the resolvent kernels $I_0(x, y; \zeta)$, $I(x, y; \zeta)$, $I'(x, y; \zeta)$ on the hyperplane $\{x_3 = 0\}$. Thus, the operator Λ defined by $\mathcal{D}(\Lambda) = \mathcal{D}(A)$, $\Lambda f = \chi_+ L_0 f + \chi_- L_\sigma f$, $f \in \mathcal{D}(\Lambda)$, is maximal dissipative [1] with adjoint $\Lambda^* = \chi_+ L_0 f + \chi_- L_\sigma^* f$, $f \in \mathcal{D}(\Lambda^*) = \mathcal{D}(\Lambda) = \mathcal{D}(A)$. It follows from the form of A [3, p. 22] that a function $f \in \mathcal{D}(A)$ if and only if $Bf(x', 0^+) = Bf(x', 0^-)$, $Bf = {}^t(f_1, f_2, f_4, f_5)$, i.e., the tangential components of the electric and magnetic fields are continuous across the interface $\{x_3 = 0\}$. From the above expressions for the free-space resolvents it is now possible to compute the trace of their kernels on $\{x_3 = 0\}$ in a manner amenable to computation of the resolvent kernels of the resolvents $G(\zeta)$, $G^*(\zeta)$ of Λ, Λ^* as the sum of "incident," "reflected," and "transmitted" waves matched on the interace $\{x_3 = 0\}$ by the condition $Bf(x', 0^+) = Bf(x', 0^-)$. While this might seem to be an incredibly "messy" business, our discovery of so-called "paramutation relations" made it possible to solve only two simple 2×2 algebraic systems to completely determine $G(\zeta)$, $G^*(\zeta)$ [3]. The expressions for them involve two algebraic functions which arise naturally in computing the traces of the free-space resolvents on the hyperplane $\{x_3 = 0\}$. They are needed below and are recorded here: for $\xi \in \mathbb{R}^2$, $|\xi| \neq 0$

$$\overset{o}{\tau}(\xi; \zeta) = c_0^{-1} \sqrt{(\zeta^2 - c_0^2 |\xi|^2)}, \quad \mathrm{Im}\, \overset{o}{\tau} \geq 0, \tag{2.7}$$

in the ζ plane with branch cuts $(-\infty, -c_0|\xi|)$, $(c_0|\xi|, +\infty)$. Similarly,

$$\tau(\xi; \zeta) = \tau_>(\xi; \zeta) + \tau_<(\xi; \zeta) = c^{-1} \sqrt{[\zeta(\zeta + i\sigma/\varepsilon) - c^2|\xi|^2]}, \ \mathrm{Im}\, \tau \geq 0, \tag{2.8}$$

where $\tau_>(\xi; \zeta)$ has branch cuts

$$(-\infty - i\sigma/2\varepsilon, \sqrt{[c^2|\xi|^2 - \sigma^2/4\varepsilon^2]} - i\sigma/2\varepsilon),$$

$$(\sqrt{[c^2|\xi|^2 - \sigma^2/4\varepsilon^2]} - i\sigma/2\varepsilon, \infty - i\sigma/2\varepsilon),$$

and $\tau_<$ has the branch cut

$$(-i\sigma/2\varepsilon - i\sqrt{[c^2|\xi|^2 - \sigma^2/4\varepsilon^2]}, -i\sigma/2\varepsilon + i\sqrt{[c^2|\xi|^2 - \sigma^2/4\varepsilon^2]})$$

see [3, p. 17]).

This construction of $G(\zeta)$ is possible only for $\zeta \notin \sigma(\Lambda)$. In addition to $\sigma(L_0)$, $\sigma(L_\sigma)$ there are three other components of $\sigma(\Lambda)$ which arise as the three zeros, $s_0(\xi)$, $s_+(\xi) = -\overline{s_-}(\xi)$, of the determinant of the algebraic systems mentioned above. These components of $\sigma(\Lambda)$ correspond to the AH and surface waves. The graphs of these zeros are shown in the following illustration of $\sigma(\Lambda)$.

[1] The dissipative operator A of [14] will here be $A = -iL$, $A^* = iL^*$ so the dissipative conditions used there here become $0 \geq \mathrm{Re}(f, Af) = \mathrm{Im}(f, Lf)$, $0 \geq \mathrm{Re}(f, A^*f) = \mathrm{Im}(L^*f, f)$.

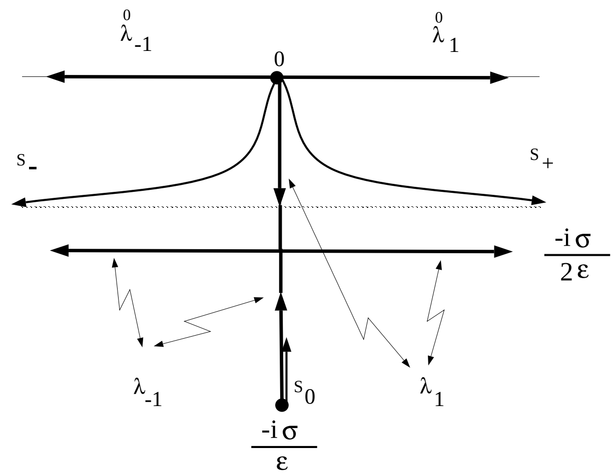

Figure: $\sigma(\Lambda)$

The solid circles indicate the embedded eigenvalues. The double vertical line indicates that this portion of the spectrum is covered twice: once by $\lambda_{-1}(\eta)$ and once by $s_0(\xi)$. The dashed line is the asymptote $-i\sigma c^2/(2\varepsilon(c^2+c_0^2))$ of $s_\pm$.

Below we shall need the properties of these zeros in a neighborhood of $|\xi| = 0$ [3, Theorem 3.4.7]:

$$
\begin{aligned}
s_0(\xi) &= -i \overset{o}{s}(\xi), \\
\overset{o}{s}(\xi) &= \sigma/\varepsilon + O(|\xi|^2), \\
\partial_{\xi_j} \overset{o}{s}(\xi) &= -(2c_0^2\varepsilon/\sigma)\xi_j + O(|\xi|^3), \\
\partial^2_{\xi_j} \overset{o}{s}(\xi) &= O(1); \\
s_\pm(\xi) &= c_0|\xi| + O(|\xi|^2), \\
\partial_{\xi_j} s_\pm(\xi) &= c_0\xi_j/|\xi| - 2i\xi_j c_0^2\varepsilon/\sigma + O(|\xi|^2), \\
\partial^2_{\xi_j} s_\pm(\xi) &= O(|\xi|^{-1}).
\end{aligned}
\tag{2.9}
$$

Correspondingly, from (2.7), (2.8) with $\overset{o}{\tau}_s = \overset{o}{\tau}(s;\xi)$, $\tau_s = \tau(s;\xi)$, $s = s_0,\ s_\pm$:

$$
\begin{aligned}
\overset{o}{\tau}_{s_0}(\xi) &= O(1), \\
\partial_{\xi_j} \overset{o}{\tau}_{s_0}(\xi) &= O(|\xi|), \\
\partial^2_{\xi_j} \overset{o}{\tau}_{s_0}(\xi) &= O(1); \\
\tau_{s_0}(\xi) &= O(|\xi|^2), \\
\partial_{\xi_j} \tau_{s_0}(\xi) &= O(|\xi|^{-1}),
\end{aligned}
$$

$$
\begin{aligned}
\partial^2_{\xi_j}\tau_{s_0}(\xi) &= O(|\xi|^{-4}); \\
\overset{o}{\tau}_{s_\pm}(\xi) &= O(|\xi|^{3/2}), \\
\partial_{\xi_j}\overset{o}{\tau}_{s_\pm}(\xi) &= O(|\xi|^{1/2}), \\
\partial^2_{\xi_j}\overset{o}{\tau}_{s_\pm}(\xi) &= O(|\xi|^{-3/2}), \\
\tau_{s_\pm}(\xi) &= O(|\xi|^{1/2}), \\
\partial_{\xi_j}\tau_{s_\pm}(\xi) &= O(|\xi|^{-1/2}), \\
\partial^2_{\xi_j}\tau_{s_\pm}(\xi) &= O(|\xi|^{-3/2}).
\end{aligned} \tag{2.10}
$$

For $\zeta \notin \sigma(\Lambda)$ we now have a representation of the resolvent $G(\zeta) = [\Lambda - \zeta I]^{-1}$, and for such ζ we thus solve problem (1.6). However, our purpose is to solve problem (1.4), and to this end the present representation of $G(\zeta)$ is unsuitable: the spectral parameter appears in many untoward places, and, in passing to the limit $\epsilon \downarrow 0$, no intelligible result can be achieved. We circumvent this state of affairs by representing the resolvent as a superposition of generalized eigenfunctions.

With $G(\zeta)$, $G^*(\zeta)$ in hand it is possible to compute the generalized eigenfunction expansion for Λ, Λ^*. For $x \in \mathbb{R}^3$ the Parseval identity for Λ , Λ^* has the form [3, (5.6.3)]

$$
f(x) = \Pi_0 f(x) + \sum_{k=s_0,s_\pm} \Sigma_k^* \Sigma_k' f(x) + \sum_{j=\pm 1} \overset{o}{\Psi}{}_j^* \overset{o}{\Psi}{}_j' f(x) + \sum_{j=\pm 1} \Psi_j^* \Psi_j' f(x). \tag{2.11}
$$

Here Π_0 is the projection onto the null space of Λ (data giving rise to static solutions), while the remaining operators Σ_k^*, Σ_k', etc. are defined by the generalized eigenfunctions of Λ, Λ^*. They are specified in subsequent sections as needed. Modulo minor technicalities, in an obvious notation the solution of problem (1.6) has the form

$$
\begin{aligned}
v(\cdot,\zeta) &= -\zeta^{-1}\Pi_0 f + \sum_{j=s_0,s_\pm} \Sigma_j^*[s_j(\cdot) - \zeta]^{-1}\Sigma_j' f + \\
&\quad \sum_{j=\pm 1} \overset{o}{\Psi}{}_j^* [\overset{o}{\lambda}_j(\cdot) - \zeta]^{-1} \overset{o}{\Psi}{}_j' f(x) + \\
&\quad \sum_{j=\pm 1} \Psi_j^*[\lambda_j(\cdot) - \zeta]^{-1}\Psi_j' f(x) \qquad (2.12) \\
&\equiv v_0(\cdot;\zeta) + v_{s_0}(\cdot;\zeta) + v_{s_+}(\cdot;\zeta) + v_{s_-}(\cdot;\zeta) + \\
&\quad \overset{o}{v}_p(\cdot;\zeta) + v_p(\cdot;\zeta).
\end{aligned}
$$

In the following sections we analyze each of these terms and their limits as $\epsilon \downarrow 0$ ($\zeta = \nu + i\epsilon,\ \nu > 0,\ \epsilon > 0$) and thus establish the validity of our theorem. It is obvious from what follows that the derivatives for the various components of v can be analyzed in exactly the same way as the components themselves, and to simplify notation we therefore restrict our consideration to the latter.

3 The Static Component of the Solution

The static component $v_0(x) = -\zeta^{-1}\Pi_0 f(x)$. From [3, (5.1.4)] with $\overset{o}{P}_0\ (\eta)$ from § 2 for $x \in \mathbb{R}^3_+$, $\mu = \mu_0$, $\eta = (\xi, \rho) \in \mathbb{R}^3$, $\xi \in \mathbb{R}^2$, $\rho \in \mathbb{R}$, $d(\xi) = (\xi_1, \xi_2, i|\xi|)$, $w(\xi, \rho) = ({}^t d(\xi)(\xi, \rho))$, and $\overline{0} = (0, 0, 0)$

$$(2\pi)^{3/2}\Pi_0 f(x) = \int \exp(ix\eta)\ \overset{o}{P}_0\ (\eta)\Phi f(\eta)\, d\eta -$$

$$\int \exp(ix'\xi - x_3|\xi|)|\eta|^{-2}\ \ {}^t\!\left(w(\xi,\rho)\Phi f(\eta), \overline{0}\right)\, d\eta$$

$$\equiv \overset{o}{P}_0\ f(x) - {}^t(I(x), 0),$$

The term $\overset{o}{P}_0\ f(x)$ is a singular integral and is $O(|x|^{-3+\mu})$, $\mu \in (0,1)$ [8, 9]. It is routine integration by parts to show that $I(x) = O(|x|^{-2})$. We may thus assert that $v_0(x) = O(|x|^{-2})$.

4 The Surface-Wave Components of the Solution

To study the components $v_{s_\pm}$ of the solution we need the representations of [3, (5.2.20), (5.2.23), (5.2.24), (5.2.34)]: with $s = s_\pm$, $x, y \in \mathbb{R}^3_+$, and $\overset{o}{\tau}_s$, τ_s of (2.7), (2.8)

$$v_s(\cdot;\zeta) = \Sigma^*_s[s(\cdot) - \zeta]^{-1}\Sigma'_s f,$$

$$\begin{aligned}
\Phi_2 v_s(\xi, x_3; \zeta) &= |\beta_s(\xi)|[s(\xi) - \zeta]^{-1}\exp(i\ \overset{o}{\tau}_s\ x_3) \times \\
&\quad \times \overset{o}{e}\ (\overset{o}{\tau}_s)\ \ {}^t\overline{\overset{o}{e}\ (-\ \overset{o}{\tau}_s)}\ \tilde{f}(\xi), \qquad (4.1)\\
\tilde{f}(\xi) &= \int \exp[-i\overline{\overset{o}{\tau}_s}(\xi)y_3]E_+\Phi_2 f(\xi, y_3)\, dy_3,\\
\overset{o}{e}\ (\overset{o}{\tau}_s) &= {}^t(-\xi_1\ \overset{o}{\tau}_s, -\xi_2\ \overset{o}{\tau}_s, |\xi|^2, \varepsilon_0 s\xi_2, -\varepsilon_0 s\xi_1, 0),\\
\beta_s(\xi) &= i\tau_s^2|\xi|^{-2}(\varepsilon\ \overset{o}{\tau}_s{}^3 + \varepsilon_0\tau_s^3 + i2^{-1}\varepsilon_0\mu\sigma s\ \overset{o}{\tau}_s)^{-1}.
\end{aligned}$$

Since $s(\xi)$, $\xi \in \mathbb{R}^2$, is always a positive distance from $\zeta = \nu + i\epsilon$, $\nu > 0$ (see Figure), the above expression contains no singularity, and, as $\Phi_2 f(\xi, y_3)$ is rapidly decreasing in ξ , the only possible impediment to integration by parts is the behavior of this expression near $\{|\xi| = 0\}$. Using (2.10), it is straightforward to verify that near $\{|\xi| = 0\}$

$$\begin{aligned}
|\beta_s(\xi)| &= O(|\xi|^{-5/2}),\\
\partial_{\xi_j}|\beta_s(\xi)| &= O(|\xi|^{-7/2}),\\
\partial^2_{\xi_j}|\beta_s(\xi)| &= O(|\xi|^{-9/2}),\\
\partial_{\xi_j}[s(\xi) - \zeta]^{-1} &= O(1), \qquad (4.2)\\
\partial^2_{\xi_j}[s(\xi) - \zeta]^{-1} &= O(|\xi|^{-1}),
\end{aligned}$$

$$
\begin{aligned}
(\overset{o}{e}\,(\overset{o}{\tau}_s)\ \ {}^t\overline{\overset{o}{e}\,(-\overset{o}{\tau}_s)})_{ik} &= O(|\xi|^4),\\
\partial_{\xi_j}(\overset{o}{e}\,(\overset{o}{\tau}_s)\ \ {}^t\overline{e(-\overset{o}{\tau}_s)})_{ik} &= O(|\xi|^3),\\
\partial^2_{\xi_j}(\overset{o}{e}\,(\overset{o}{\tau}_s)\ \ {}^t\overline{e(-\overset{o}{\tau}_s)})_{ik} &= O(|\xi|^{3/2}),\\
i,\ k = 1, ..., 6.
\end{aligned}
$$

Now it follows from (2.7), (2.9) that there exist constants c, $d > 0$ such that for $|\xi| < d < 1$, Im $\overset{o}{\tau}(\xi;\zeta) > c|\xi|^{3/2}$. Thus, from (4.1), (4.2) for a constant $b = b(\nu, f) > 0$

$$
\begin{aligned}
x_3^2|v(x;\nu)| &< bx_3 \int \exp(-c|\xi|^{3/2}x_3)|\xi|^{5/2}\, d|\xi| + O(1)\\
&< bx_3^2 \int_{|\xi|\le d} \exp(-c|\xi|^{3/2}x_3)|\xi|^2\, d|\xi| + O(1)\\
&= -(bx_3^2/c)\partial_3 \int_{|\xi|\le d} \exp(-c|\xi|^{3/2}x_3)|\xi|^{1/2}\, d|\xi| + O(1)\\
&= -(2bx_3^2/3c^2)\partial_3 \int_{|\xi|\le d} \exp(-c|\xi|^{3/2}x_3)\, d(c|\xi|^{3/2}) + O(1)\\
&= (2bx_3^2/3c)\partial_3([\exp(-cd^{3/2}x_3) - 1]/x_3) + O(1)\\
&= (2b/3c)[1 - \exp(-cd^{3/2}x_3)] + O(1)\\
&\le C(\nu, f) < \infty.
\end{aligned}
\tag{4.3}
$$

To estimate $|x'|^2 v_s(x;\nu)$ we compute

$$
\begin{aligned}
\partial^2_{\xi_j} v_s(x;\nu) &= G\partial^2_{\xi_j}|\beta_s|^2 + 2|\beta_s|^2\partial_{\xi_j}G + |\beta_s|^2\partial^2_{\xi_j}G\\
&= O(|\xi|^{-1}) + x_3O(1) + x_3^2O(|\xi|^{7/4}).
\end{aligned}
\tag{4.4}
$$

The x_3^2-term can be estimated as in (4.3). For the x_3-term we have with $c = c(\nu, f)$

$$
\begin{aligned}
x_3O(1) &\le cx_3 \int_{|\xi|\le d} \exp(-c|\xi|^{3/2}x_3)|\xi|\, d|\xi|\\
&\le (cx_3/2)\int_0^\infty \exp(-c|\xi|^2x_3)\, d|\xi|^2\\
&\le C(f, \nu) < \infty.
\end{aligned}
\tag{4.5}
$$

Thus, from (4.4), (4.5) we have

$$
|x'|^2|v_s(x;\nu)| \le C(f, \nu) < \infty, \tag{4.6}
$$

and from (4.1), (4.3), (4.6) we conclude that $v_s(x;\nu) = O(|x|^{-2})$.

5 The AH-Wave Component of the Solution

The component v_{s_0} is given by (4.1) with $s = s_\pm$ replaced by s_0. Again, the only impediment to integration by parts is the behavior of $\Phi_2 v_{s_0}(\xi, x_3)$ and its derivatives near $|\xi| = 0$. From (2.10) in place of (4.2) we now obtain

$$\begin{aligned}
|\beta_{s_0}(\xi)| &= O(|\xi|^2), \\
\partial_{\xi_j}|\beta_{s_0}(\xi)| &= O(|\xi|^{-1}), \\
\partial^2_{\xi_j}|\beta_{s_0}(\xi)| &= O(|\xi|^{-4}), \\
\left[\overset{o}{e}(\overset{o}{\tau}_{s_0})(\overline{{}^t\overset{o}{e}(-\overset{o}{\tau}_{s_0})})\right]_{ik} &= O(|\xi|^2), \\
\partial_{\xi_j}\left[\overset{o}{e}(\overset{o}{\tau}_{s_0})(\overline{{}^t e(-\overset{o}{\tau}_{s})})\right]_{ik} &= O(|\xi|),\ i,\ k = 1, ..., 6.
\end{aligned} \tag{5.1}$$

Because of the third and fourth estimates, we see that $\partial^2_{\xi_j} v_{s_0}(\xi, x_3) = O(|\xi|^{-2})$ which, on taking Φ_2^*, gives a nonconvergent integral, so that the best we can do is to integrate by parts once in the tangential directions. Since by (2.7), (2.9), Im $\overset{o}{\tau}_{s_0} = |\overset{o}{\tau}_{s_0}| \geq b > 0$, the decay in the x_3-direction is exponential for all x'. Further,

$$\partial_{\xi_j}\Phi_2 v_{s_0}(\xi, x_3; \nu) = \exp(i\overset{o}{\tau}_{s_0} x_3)h(\xi) + x_3\exp(i\overset{o}{\tau}_{s_0} x_3)k(\xi)$$

with $h,\ k \in L_1(\mathbb{R}^2)$. By the Riemann-Lebesgue lemma we thus have $v_{s_0}(x; \nu) = o_{x_0}(|x'|^{-1})$ for all $x_3 > 0$. To eliminate the x_0 in o_{x_0} a slight modification of the proof of the Riemann-Lebesgue lemma is required (see, e. g., [2]). The asymptotic behavior of v_{s_0} is thus as asserted in the theorem.

6 The Spatial Wave $\overset{o}{v}_p(x; \zeta)$

From [3, (5.4.14), (5.4.14'), (5.4.16)] for $f \in C_0^\infty(\mathbb{R}^3_+; \mathbb{C}^6)$, $E \equiv E_+$, $\zeta = \nu + i\epsilon$, $\eta = |\eta|s$, $s \in S^2$, $\tilde{\eta} = (\xi, -\rho)$, $x \in \mathbb{R}^3_+$, $x = |x|\omega$, $\omega \in S^2$, $j = \pm 1$

$$\begin{aligned}
\overset{o}{v}_p(x; \zeta) &= \overset{o}{v}{}_p^{(1)}(x; \zeta) + \overset{o}{v}{}_p^{(-1)}(x; \zeta), \\
(2\pi)^{3/2} v_p^{(j)}(x; \zeta) &= (2\pi)^{3/2}\left(\overset{o}{\Psi}{}^*_j\right)_\pm [\overset{o}{\lambda}_j(\eta) - \zeta]^{-1}\left(\overset{o}{\Psi}{}'_j\right)_\pm f(x) \\
&= \int \chi_\pm(s_3)[\overset{o}{\lambda}_j(\eta) - \zeta]^{-1}\{\exp(ix\eta)\overset{o}{P}_j \Phi f(\eta) \\
&\quad + \exp(ix\tilde{\eta})C_{0,j}(\eta)\overset{o}{P}_j(\omega)C_{0,j}(\tilde{\eta})\Phi f(\tilde{\eta}) \\
&\quad - \exp(ix\tilde{\eta})C_{0,j}(\eta)\overset{o}{P}_j(\omega)\Phi f(\eta) \\
&\quad - \exp(ix\eta)\overset{o}{P}_j(\omega)C_{0,j}(\tilde{\eta})\Phi f(\tilde{\eta})\}\, d\eta,
\end{aligned} \tag{6.1}$$

where the $\overset{o}{P}_j$ are given in §2 and $C_{0,j}$ is given in (6.3). From the paramutation relations [3, p. 131] for the reflection terms we have

$$C_{0,j}(\eta)\overset{o}{P}_j(\omega)=\overset{o}{P}_j(\widetilde{\omega})C_{0,j}(\eta),$$

and hence the last two terms are

$$\begin{aligned}-\int \chi_\pm(s_3)[\overset{o}{\lambda}_j(\eta)-\zeta]^{-1}[\exp(ix\tilde{\eta})\overset{o}{P}_j(\widetilde{\omega})C_{0,j}(\eta)\Phi f(\eta)+\\ \exp(ix\eta)\overset{o}{P}_j(\omega)C_{0,j}(\tilde{\eta})\Phi f(\tilde{\eta})]\,d\eta= \\ -\int[\overset{o}{\lambda}_j(\eta)-\zeta]^{-1}[\exp(ix\tilde{\eta})\overset{o}{P}_j(\tilde{\omega})C_{0,j}(\eta)\Phi f(\eta)\,d\eta.\end{aligned}\tag{6.2}$$

Again, the paramutation relations [3, p. 131] give

$$C_{0,j}(\eta)\overset{o}{P}_j(\omega)C_{0,j}(\tilde{\eta})=\overset{o}{P}_j(\omega)C_{0,j}(\eta)C_{0,j}(\tilde{\eta}),$$

and it is a fact that $C_{0,j}(\eta)C_{0,j}(\tilde{\eta})=I$, as we now show. First, from [3, (5.4.15)] with $s=(s',s_3)$, $\eta=|\eta|s$

$$\begin{aligned}C_{0,j}(\eta)&=\left[\overset{o\,j}{\Delta}_e(\tilde{\eta})/\overset{o\,j}{\Delta}_e(\eta)\right]Q+\left\{2\ell_j(\eta)/[\overset{o\,j}{\Delta}_e(\eta)\overset{o\,j}{\Delta}_m(\eta)]\right\}H(s),\\ Q&=\mathrm{diag}\,[1,1,-1,-1,-1,1],\\ H(s)&=\begin{bmatrix} h(s_1,s_2)-h_0 & \underline{O}\\ \underline{O} & -h(s_2,-s_1)\end{bmatrix},\\ h(a,b)&=\begin{bmatrix} a^2 & ab & 0\\ ab & b^2 & 0\\ 0&0&0\end{bmatrix},\\ h_0&=\mathrm{diag}\,(0,0,|s'|^2)\\ \ell_j(\eta)&=c_0^{-2}-c^2-ij\sigma\mu c_0^{-1}|\eta|^{-1},\end{aligned}\tag{6.3}$$

and, concerning the scalar functions $\ell_j(\eta)$, $\overset{o\,j}{\Delta}_e(\eta)$, $\overset{o\,j}{\Delta}_m(\eta)$, we need know only that they satisfy the following identity [3, p. 131]:

$$\overset{o\,j}{\Delta}_e(\tilde{\eta})\overset{o\,j}{\Delta}_m(\eta)+\overset{o\,j}{\Delta}_e(\eta)\overset{o\,j}{\Delta}_m(\tilde{\eta})=-2\ell_j(\eta)|s'|^2.\tag{6.4}$$

Using the facts that $QH(s')=H(s')Q$ and $H^2(s')=|s'|^2QH(s')$ as well as the identity (6.4), from (6.3) it is now straightforward to compute $C_{0,j}(\eta)C_{0,j}(\tilde{\eta})=I$.

From (6.1), (6.2) we now have

$$
\begin{aligned}
(2\pi)^{3/2} \overset{o}{v}_p (x;\eta) &= \int_{\mathbb{R}^3} [\overset{o}{\lambda}_j (\eta) - \zeta]^{-1} \{ \\
&\quad \chi_\pm(s_3) \exp(ix\eta) \overset{o}{P}_j (\omega) \Phi f(\eta) + \\
&\quad \chi_\pm(s_3) \exp(ix\tilde{\eta}) \overset{o}{P}_j (\tilde{\omega}) \Phi f(\tilde{\eta}) - \\
&\quad \exp(ix\tilde{\eta}) \overset{o}{P}_j (\tilde{\omega}) C_{0,j}(\eta) \Phi f(\eta)\}\, d\eta = \\
&= \int_{\mathbb{R}^3} [\overset{o}{\lambda}_j (\eta) - \zeta]^{-1} [\exp(ix\eta) I - \qquad (6.5) \\
&\quad \exp(ix\tilde{\eta}) C_{0,j}(\eta)] \overset{o}{P}_j (\omega) \Phi f(\eta)\, d\eta \\
&\equiv \int_{\mathbb{R}^3} [\overset{o}{\lambda}_j (\eta) - \zeta]^{-1} \exp(ix\eta) \tilde{f}_j(\eta)\, d\eta,
\end{aligned}
$$

$$
\tilde{f}_j(\eta) = \overset{o}{P}_j (s) \left[\Phi f(\eta) - C_{0,j}(\tilde{\eta}) \Phi f(\tilde{\eta})\right].
$$

Now the last expression is exactly of the same form as (2.10), (2.23) of [9]. The proof of the existence of $\overset{o}{v}{}_p^{(j)} (x;\nu) = \overset{o}{v}{}_p^{(j)} (x;\nu + i0)$ and the estimate of $\overset{o}{v}{}_p^{(1)} (x;\nu)$ for large $|x|$ by stationary phase and of $\overset{o}{v}{}_p^{(-1)} (x;\nu)$ by parts integration is exactly the same as there and will not be reproduced here. The result for $\overset{o}{v}{}_p^{(1)} (x;\nu)$ is as stated for $\overset{o}{v}_p (x)$ in the theorem, while $\overset{o}{v}{}_p^{(-1)} (x;\nu) = O(|x|^{-2})$.

7 The Spatial Wave $v_p(x;\zeta)$

From [3, (5.5.10), (5.5.14)] with $\lambda_j(\eta)$, $P_j(\eta)$ of § 2, and $\tilde{\eta} = (\eta_1, \eta_2, -\eta_3)$ for $x \in \mathbb{R}^3$, $f \in C_0^\infty(\mathbb{R}^3_+; \mathbb{C}^6)$

$$
\begin{aligned}
v_p(x;\zeta) &= \Psi_1^*[\lambda_1(\cdot) - \zeta]^{-1} \Psi_1' f(x) \\
&+ \Psi_{-1}^*[\lambda_{-1}(\cdot) - \zeta]^{-1} \Psi_{-1}' f(x) \\
&= (2\pi)^{-3/2} \int \chi_\pm(s_3) \{\exp(ix\gamma^1) D_1(\eta) P_1(\eta) \times \qquad (7.1) \\
&\quad D_1(\tilde{\eta})[\lambda_1(\eta) - \zeta]^{-1} \Phi f(\overline{\gamma}^1) + \exp(ix\gamma^{(-1)}) \times \\
&\quad D_{-1}(\eta) P_{-1}(\eta) D_{-1}(\tilde{\eta}) [\lambda_{-1}(\eta) - \zeta]^{-1} \Phi f(\overline{\gamma}^{(-1)})\}\, d\eta,
\end{aligned}
$$

where the matrices D_j are given in [3, (5.5.11)], and from [3, pp. 162, 166, 153]

$$
\gamma^j = (\xi, \gamma_3^j), \quad \gamma_3^j = n^{-1} |\eta| \varphi_3^j, \qquad (7.2)
$$

and for $c|\eta| > \sigma/2\varepsilon$

$$2^{1/2} c|\eta| \operatorname{Im} \varphi_3^j = \left\{ \sqrt{[(\widetilde{\lambda}^2\beta - \widetilde{\beta})^2 + \sigma^2\widetilde{\lambda}/\varepsilon^2]} + \widetilde{\lambda}^2\beta - \widetilde{\beta} \right\}^{1/2}$$

$$2^{1/2} c|\eta| \operatorname{Re} \varphi_3^j = \left\{ \sqrt{[(\widetilde{\lambda}^2\beta - \widetilde{\beta})^2 + \sigma^2\widetilde{\lambda}/\varepsilon^2]} - \widetilde{\lambda}^2\beta + \widetilde{\beta} \right\}^{1/2}$$

$$\beta = 1 - n^2|\omega'|^2, \quad \tilde{\beta} = \sigma^2(1 + n^2|\omega'|^2)/4\varepsilon^2 \tag{7.3}$$

$$\widetilde{\lambda} = |\lambda| = \left| c^2|\eta|^2 - \sigma^2/4\varepsilon^2 \right|^{1/2},$$

while for $c|\eta| < \sigma/2\varepsilon$

$$\begin{aligned} c|\eta| \operatorname{Im} \varphi_3^j &= |\lambda_j(\eta) \pm c_0|\xi||, \\ \operatorname{Re} \varphi_3^j &= 0. \end{aligned} \tag{7.4}$$

From (7.3) as $c|\eta| \to \sigma/2\varepsilon + 0$, $c|\eta| \operatorname{Re} \varphi_3^j \to 0$, while as $c|\eta| \to \sigma/2\varepsilon \pm 0$, $c|\eta| \operatorname{Im} \varphi_3^j \to \widetilde{\beta}^{1/2} = (\sigma/2\varepsilon)(1+n^2|\omega'|^2)^{1/2}$. We choose a neighborhood N_δ of the sphere $\{c|\eta| = \sigma/2\varepsilon\}$ such that on N_δ $c|\eta| \operatorname{Im} \varphi_3^j > \sigma/4\varepsilon$, i.e., $\operatorname{Im} \gamma_3^j > \sigma/4\varepsilon c_0$. Let $\psi \in C_0^\infty(\mathbb{R}^3)$ be such that $\psi|_{N_{\delta/2}} = 1$, supp $\psi \subset N_\delta$. Introducing $1 = \psi + (1 - \psi)$ into (7.1), we have in an obvious notation

$$v_p = v_p^\delta + v_p^< + v_p^>, \tag{7.5}$$

where for $v_p^<$ ($v_p^>$) the support of the integrand is contained in $\{c|\eta| < \sigma/2\varepsilon\} \cap N_{\delta/2}$ ($\{c|\eta| > \sigma/2\varepsilon\} \cap N_{\delta/2}$). We consider each term of (7.5) separately.

In N_δ, up to terms of $O(|\lambda(\eta)|)$,

$$\lambda_1(\eta) = \lambda_{-1}(\eta),$$

$$\gamma^1(\eta) = \gamma^{(-1)}(\eta),$$

$$D_1(\eta) = D_{-1}(\eta), \; D_1(\widetilde{\eta}) = D_{-1}(\tilde{\eta}),$$

so that the integrand of v_p^δ is

$$\chi_\pm(s_3) \exp(ix\gamma^1) D_1(\eta)[P_1(\eta) + P_{-1}(\eta)] D_1(\widetilde{\eta})[\lambda_1(\eta) - \zeta]^{-1} \Phi f(\overline{\gamma}^1) \tag{7.6}$$

plus a term regular at $\lambda(\eta) = 0$,

Since $P_1(\eta) + P_{-1}(\eta)$ is not singular at $\lambda(\eta) = 0$ (see § 2), for any $x' \in \mathbb{R}^2$

$$|v_p^\delta(x;\nu)| < \text{const } \exp(-x_3\sigma/4\varepsilon c_0),$$

while integration by parts in the tangential variables gives for any x_3

$$|x'|^2 |v_p^\delta(x;\nu)| \leq \text{ const.}$$

Thus,

$$|v_p^\delta(x;\nu)| = O(|x|^{-2}). \tag{7.7}$$

We remark that the full details of writing the integrand of v_p^δ in the form (7.6) are very tedious and are given in [3].

We need be concerned with the integrand of $v_p^<$ only in a neighborhood of $|\eta| = 0$. From (7.4) for $|\eta|$ near 0

$$\gamma_3^1(\eta) \sim |\xi|,\ \gamma_3^{(-1)}(\eta) \sim (\sigma^2/\varepsilon^2 + c_0|\xi|^2)^{1/2}/c_0.$$

Thus, for any x' the (-1)-term of v_p is less than or equal a constant times $\exp(-x_3\sigma\varepsilon/c_0)$, while the 1-term is less than or equal to a constant divided by x_3^2. Integration by parts in the tangential variables now gives

$$v_p^<(x;\nu) = O(|x|^{-2}). \tag{7.8}$$

For $v_p^>$ only the behavior of the integrand as $|\eta| \to \infty$ signifies. Here $\gamma_3^j = x_j + iy$, and from [3, (5.5.55)]

$$y(\eta) = (|\eta|/n\sqrt{2})\left\{|\beta| - \beta + (a_0/2|\eta|^2)[(1-\beta)\int_0^1 d^{-1/2}(t|\eta|,\beta)\,dt + 1]\right\}^{1/2}$$

$$d(t|\eta|,\beta) = \beta^2 + a_0 t|\eta|^{-2}(1-\beta).$$

In both cases $\beta < 0$ and $\beta > 0$ we have $y(\eta) > a_0^{1/2}/2n$, so that for any x' $\quad v^>(x;\nu) \leq$ const $\exp(-x_3 a_0^{1/2}/2n)$, and integration by parts on ξ' then gives

$$|v^>(x;\nu)| = O(|x|^{-2}). \tag{7.9}$$

To take care of the singularity of $D_j(\eta)$ at ∞, occuring due to the factor Δ_m^j [3, (5.5.8), (5.5.11)], we multiply the integrand of $v_p^>(x;\nu)$ by $1 = |\xi|^2|\xi|^{-2}$, absorb $|\xi|^2$ into $\Phi f(\overline{\gamma}^j)$, $j = -1, 1$, which remains rapidly decreasing in ξ, and we then have $|\xi|^2|\Delta_m^j|^2 \geq$ const > 0 at ∞. The expressions (7.7), (7.8), (7.9) now yield the estimate of the theorem.

8 Concluding Remarks

Point sources such as dipoles, etc. are dear to the hearts of applied people, and it might be thought that for such singular sources the resulting field could be different from that described in the theorem. This is not the case. As we have pointed out previously [9], the fields due to such sources can be obtained simply as appropriate specializations of the Green function, and the asymptotic behavior of the Green function can be read off from the results presented above. There is thus no hope for the ghost-chasers who are fain to discover new mysteries concerning surface waves.

References

[1] A. Baños, Dipole Radiation in the Case of a Conducting Half Space, Pergammon Press, Oxford (1966).

[2] H. Dym and H. McKean, Fourier Series and Integrals, Academic Press, N. Y. (1972).

[3] D. S. Gilliam and J. R. Schulenberger, The Propagation of Electro- magnetic Waves Through, Along, and Over a Three-Dimensional Conducting Half Space, Methoden und Verfahren der mathematischen Physik, Verlag Peter Lang, Frankfurt am Main (1986).

[4] D. S. Gilliam and J. R. Schulenberger, "Spectral analysis of a dissipative problem in electrodynamics: the Sommerfeld problem," *Acta Applicandae Mathematicae*, vol. 6, , 63-94 (1986).

[5] D. S. Gilliam and J. R. Schulenberger, "On the structure of surface waves," *Adv. Appl. Math.*, vol. 4, , 212-243 (1983).

[6] J. Loeb, "Sur les ondes de surface en régime transitoire," in: Electromagnetic Wave Theory (J. Brown, Ed.), Part 1, Pergammon Press (1967).

[7] P. Kossey et al., "Some aspects of long wave propagation," Preprint, Propagation Branch, Rome Air Development Center.

[8] S. G. Mikhlin, Multidimensional Singular Integrals and Integral Equations, Pergamon Press, Oxford (1965).

[9] J. R. Schulenberger, "Time-harmonic solutions of some dissipative problems for Maxwell's equations in a three-dimensional half space," *Pacific J. Math.*, vol. 142, 313-345, No. 2 (1990).

[10] J. R. Schulenberger, "Elastic waves in a three-dimensional half space. The Lamb problem," Report No. 12, AN AH Corp. (1990).

[11] A. Sommerfeld, "Über die Ausbreitung der Wellen in der drahtlosen Telegraphie," *Ann. Physik*, vol. 28, , No. 4, 665-736 (1909).

[12] J. Stratton, Electromagnetic Theory, McGraw-Hill, N. Y. (1941).

[13] H. Weyl, "Ausbreitung elektromagnetischer Wellen über einem ebenen Leiter," *Ann. Physik*, vol. 60, , 481-500 (1919).

[14] J. Goldstein, Semigroups of Linear Operators and Applications, Oxford University Press, New York (1985).