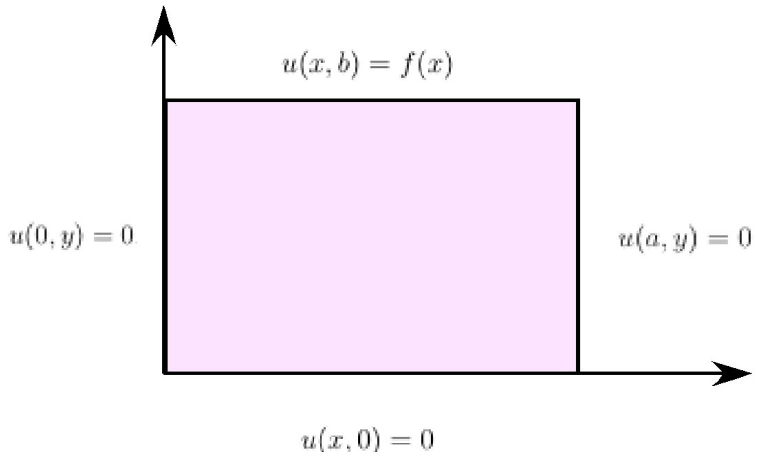
Elliptic Boundary Value Problems

In these notes we are concerned with application of the method of separation of variables applied to elliptic problems. We are mostly concerned with 2-Dimensional problems in a rectangle or a disc or annulus. These problems arise, for example, in the steady state analysis or heat and wave problems where we are interested in problems with no time dependence, so either u_t or u_{tt} is zero.

Thus we consider

$$\begin{aligned} &u_{xx}(x,y) + u_{yy}(x,y) = 0, \quad (x,y) \in [0,a] \times [0,b], \qquad (1)\\ &\quad u(0,y) = 0, \quad u(a,y) = 0\\ &\quad u(x,0) = 0, \quad u(x,b) = f(x) \end{aligned}$$



Look for simple solutions in the form

$$u(x,y) = X(x)Y(y).$$

Substituting into (1) and dividing both sides by $X(x)Y(y)$ gives

$$\frac{-Y''(y)}{Y(y)} = \frac{X''(x)}{X(x)}$$

Since the left side is independent of y and the right side is independent of x, it follows that the expression must be a constant:

$$\frac{-Y''(y)}{Y(y)} = \frac{X''(x)}{X(x)} = \lambda.$$

(Here Y' means the derivative of y with respect to y and X' means means the derivative of X with respect to x.) We seek to find all possible constants λ and the corresponding nonzero functions X and Y. We obtain

$$X'' - \lambda X = 0, \qquad Y'' + Y = 0.$$

Furthermore, the boundary conditions give

$$X(0)Y(y) = 0, \quad X(a)Y(y) = 0 \quad \text{for all } y.$$

Since $Y(y)$ is not identically zero we obtain the desired eigenvalue problem

$$X''(x) - \lambda X(x) = 0, \quad X(0) = 0, \quad X(a) = 0. \tag{2}$$

We have solved this problem many times and we have $\lambda = -\mu^2$ so that

$$X(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x).$$

Applying the boundary conditions we have

$$0 = X(0) = c_1 \Rightarrow c_1 = 0 \quad 0 = X(a) = c_2 \sin(\mu a).$$

From this we conclude $\sin(\mu a) = 0$ which implies

$$\mu = \frac{n\pi}{a}$$

and therefore

$$\lambda_n = -\mu_n^2 = -\left(\frac{n\pi}{a}\right)^2, \quad X_n(x) = \sin(\mu_n x), \quad n = 1, 2, \cdots .. \tag{3}$$

The solution of $Y'' - \left(\frac{n\pi}{a}\right)^2 Y = 0$ with $Y(0) = 0$ is then

$$Y(y) = c_1 \cosh\left(\frac{n\pi y}{a}\right) + c_2 \sinh\left(\frac{n\pi y}{a}\right) \tag{4}$$

where c_1 and c_2 are arbitrary constants. The boundary condition $Y(0) = 0$ implies

$$Y(y) = \sinh\left(\frac{n\pi y}{a}\right).$$

So we look for u as an infinite sum

$$u(x, y) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right) \tag{5}$$

The only problem remaining is to somehow pick the constants a_n and b_n so that the initial condition $u(x, b) = f(x)$ is satisfied.

Setting $y = b$ in (5), we seek to obtain $\{a_n\}$ satisfying

$$f(x) = u(x, b) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi b}{a}\right).$$

This is almost a Sine expansion of the function $f(x)$ on the interval $(0, a)$. In particular we obtain

$$\sinh\left(\frac{n\pi b}{a}\right) a_n = \frac{2}{a} \int_0^a f(x) \sin\left(\frac{n\pi x}{a}\right)\, dx. \tag{6}$$

Example 1. Consider the problem (1) with

$$f(x) = \begin{cases} x & 0 \le x \le a/2, \\ (\pi - x) & a/2 \le x \le a. \end{cases}$$

For this example (5) becomes

$$u(x,y) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right).$$

In this case we have

$$\sinh\left(\frac{n\pi b}{a}\right) a_n = \frac{2}{a}\left[\int_0^{a/2} x \sin\left(\frac{n\pi x}{a}\right)\, dx + \int_{a/2}^{a} (a-x) \sin\left(\frac{n\pi x}{a}\right)\, dx\right]$$

$$= \begin{cases} 0 & n = 2, 4, \cdots \\ \dfrac{4a}{n^2\pi^2} \sin\left(\dfrac{n\pi}{2}\right) & n = 1, 3, \cdots \end{cases}$$

We carry out the two integration by parts separately:

$$\begin{aligned} \int_0^{a/2} x \sin\left(\frac{n\pi x}{a}\right)\, dx &= \int_0^{a/2} x \left(-\frac{a}{n\pi}\cos\left(\frac{n\pi x}{a}\right)\right)'\, dx \\ &= x\left(-\frac{a}{n\pi}\cos\left(\frac{n\pi x}{a}\right)\right)\Bigg|_0^{a/2} - \int_0^{a/2} \left(-\frac{a}{n\pi}\cos\left(\frac{n\pi x}{a}\right)\right) dx \\ &= a/2\left(-\frac{a}{n\pi}\cos\left(\frac{n\pi}{2}\right)\right) + \frac{a}{n\pi}\int_0^{a/2} \cos\left(\frac{n\pi x}{a}\right)\, dx \\ &= -\frac{a^2}{2n\pi}\cos\left(\frac{n\pi}{2}\right) + \left(\frac{a}{n\pi}\right)^2 \sin\left(\frac{n\pi x}{a}\right)\Bigg|_0^{a/2} \\ &= -\frac{a^2}{2n\pi}\cos\left(\frac{n\pi}{2}\right) + \left(\frac{a}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}\right). \end{aligned}$$

and

$$\begin{aligned}\int_{a/2}^{a}(a-x)\sin\left(\frac{n\pi x}{a}\right)\,dx &= \int_{a/2}^{a}(a-x)\left(-\frac{a}{n\pi}\cos\left(\frac{n\pi x}{a}\right)\right)'\,dx\\
&= (a-x)\left(-\frac{a}{n\pi}\cos\left(\frac{n\pi x}{a}\right)\right)\Bigg|_{a/2}^{a} - \int_{a/2}^{a}(-1)\left(-\frac{a}{n\pi}\cos\left(\frac{n\pi x}{a}\right)\right)\,dx\\
&= a/2\left(\frac{a}{n\pi}\cos\left(\frac{n\pi}{2}\right)\right) - \frac{a}{n\pi}\int_{a/2}^{a}\cos\left(\frac{n\pi x}{a}\right)\,dx\\
&= \frac{a^2}{2n\pi}\cos\left(\frac{n\pi}{2}\right) - \left(\frac{a}{n\pi}\right)^2\sin\left(\frac{n\pi x}{a}\right)\Bigg|_{a/2}^{a}\\
&= \frac{a^2}{2n\pi}\cos\left(\frac{n\pi}{2}\right) + \left(\frac{a}{n\pi}\right)^2\sin\left(\frac{n\pi}{2}\right).\end{aligned}$$

Thus we have

$$\sinh\left(\frac{n\pi b}{a}\right)a_n = \frac{2}{a}\left[\left(\frac{a}{n\pi}\right)^2\sin\left(\frac{n\pi}{2}\right) + \left(\frac{a}{n\pi}\right)^2\sin\left(\frac{n\pi}{2}\right)\right]$$

or finally,

$$a_n = \frac{4a\sin\left(\frac{n\pi}{2}\right)}{n^2\pi^2\sinh\left(\frac{n\pi b}{a}\right)}.$$

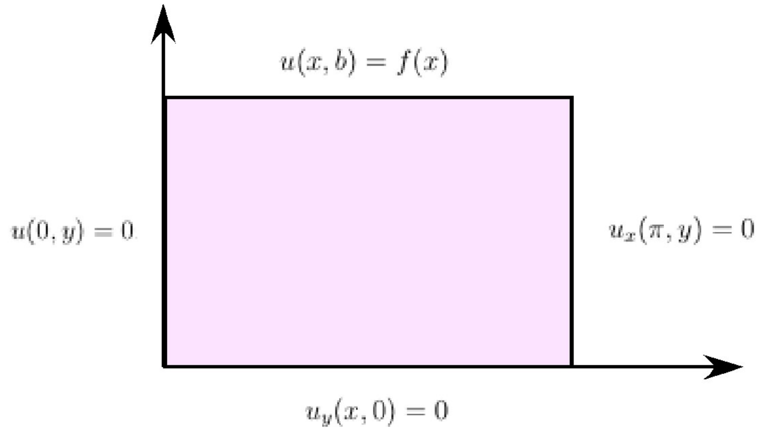
We arrive at the solution

$$u(x,y) = \frac{4a}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin\left(\dfrac{n\pi}{2}\right)}{n^2\pi^2}\frac{\sinh\left(\dfrac{n\pi y}{a}\right)}{\sinh\left(\dfrac{n\pi b}{a}\right)}\sin\left(\frac{n\pi x}{a}\right).$$

The Laplace Equation with other Boundary Conditions

Next we consider a slightly different problem involving a mixture of Dirichlet and Neumann boundary conditions. To simplify the problem a bit we set $a=\pi$ Namely we consider

$$\begin{aligned}&u_{xx}(x,y)+u_{yy}(x,y)=0,\quad (x,y)\in[0,\pi]\times[0,b],\\
&u(0,y)=0,\quad u_x(\pi,y)=0\\
&u_y(x,0)=0,\quad u(x,b)=f(x)\end{aligned}\tag{7}$$



Look for simple solutions in the form

$$u(x,y) = X(x)Y(y).$$

Substituting into (7) and dividing both sides by $X(x)Y(y)$ gives

$$\frac{-Y''(y)}{Y(y)} = \frac{X''(x)}{X(x)}$$

Since the left side is independent of y and the right side is independent of x, it follows that the expression must be a constant:

$$\frac{-Y''(y)}{Y(y)} = \frac{X''(x)}{X(x)} = \lambda.$$

(Here Y' means the derivative of y with respect to y and X' means means the derivative of X with respect to x.) We seek to find all possible constants λ and the corresponding <u>nonzero</u> functions X and Y. We obtain

$$X'' - \lambda X = 0, \qquad Y'' + Y = 0.$$

Furthermore, the boundary conditions give

$$X(0)Y(y) = 0, \quad X'(\pi)Y(y) = 0 \quad \text{for all } y.$$

Since $Y(y)$ is not identically zero we obtain the desired eigenvalue problem

$$X''(x) - \lambda X(x) = 0, \quad X(0) = 0, \quad X'(\pi) = 0. \tag{8}$$

We have solved this problem many times and we have $\lambda = -\mu^2$ so that

$$X(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x).$$

Applying the boundary conditions we have

$$0 = X(0) = c_1 \Rightarrow c_1 = 0 \quad 0 = X'(\pi) = c_2 \mu \cos(\mu \pi).$$

From this we conclude $\sin(\mu\pi) = 0$ which implies

$$\mu_n = \frac{(2n-1)}{2}$$

and therefore

$$\lambda_n = -\mu_n^2 = -\left(\frac{(2n-1)}{2}\right)^2, \quad X_n(x) = \sin(\mu_n x), \quad n = 1, 2, \cdots .. \tag{9}$$

The solution of $Y'' - (\mu_n)^2 Y = 0$ with $Y'(0) = 0$ is then

$$Y(y) = c_1 \cosh(\mu_n y) + c_2 \sinh(\mu_n y) \tag{10}$$

where c_1 and c_2 are arbitrary constants. The boundary condition $Y'(0) = 0$ implies

$$Y(y) = \cosh(\mu_n y).$$

So we look for u as an infinite sum

$$u(x, y) = \sum_{n=1}^{\infty} a_n \sin(\mu_n x) \cosh(\mu_n y) \tag{11}$$

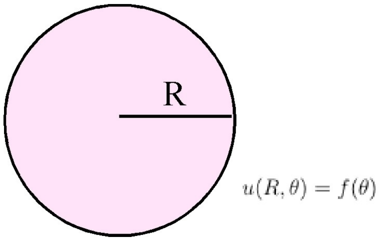
Finally we need to find the constants a_n so that

$$f(x) = u(x, b) = \sum_{n=1}^{\infty} a_n \sin(\mu_n x) \cosh(\mu_n b).$$

The Problem Problem in a Disk

Next we consider a problem in a different geometry. In particular, we look at Laplace's equation in a disk. Consider the problem

$$\begin{aligned} u_{xx}(x, y) + u_{yy}(x, y) = 0, \quad x^2 + y^2 < R^2 \qquad (12)\\ u(x, y) = f(x, y), \quad x^2 + y^2 = R^2, \\ u(x, y) \text{ bounded on } x^2 + y^2 \le R^2. \qquad (13)\end{aligned}$$



It turns out that we cannot solve this problem using separation of variables as it is written. But, as you will see, if we change coordinates to polar coordinates then separation of variables works fine.

To this end recall that polar coordinates are given by

$$x = r\cos(\theta), \quad y = r\sin(\theta)$$

or

$$r^2 = x^2 + y^2, \quad \theta = \tan^{-1}(y/x).$$

So we need to translate

$$u_{xx}(x,y) + u_{yy}(x,y) = 0$$

into the variables r and θ. First we use $r^2 = x^2 + y^2$ and implicit differentiation to compute

$$2rr_x = 2x \ \Rightarrow \ r_x = \frac{x}{r}, \quad 2rr_y = 2y \ \Rightarrow \ r_y = \frac{y}{r}$$

so we have

$$r_x = \frac{x}{r} = \cos(\theta) \ \text{ and } \ r_y = \frac{y}{r} = \sin(\theta). \tag{14}$$

Similarly, differentiating $y = r\sin(\theta)$ with respect to x and using (14) we have

$$\begin{aligned} 0 &= r_x\sin(\theta) + r\cos(\theta)\theta_x &(15)\\ &= \cos(\theta)\sin(\theta) + r\cos(\theta)\theta_x. &(16)\end{aligned}$$

Therefore

$$\theta_x = -\frac{\sin(\theta)}{r}. \tag{17}$$

Differentiating $x = r\cos(\theta)$ with respect to y and using (14) we have

$$\begin{aligned} 0 &= r_y\cos(\theta) - r\sin(\theta)\theta_y &(18)\\ &= \cos(\theta)\sin(\theta) - r\sin(\theta)\theta_y. &(19)\end{aligned}$$

Therefore

$$\theta_y = \frac{\cos(\theta)}{r}. \tag{20}$$

Nest, using the chain rule and using (14) we compute

$$u_x = u_r r_x + u_\theta \theta_x = u_r\cos(\theta) - u_\theta\frac{\sin(\theta)}{r} \tag{21}$$

and

$$u_y = u_r r_y + u_\theta \theta_y u_r\sin(\theta) + u_\theta\frac{\cos(\theta)}{r}. \tag{22}$$

Using the formulas (21) and (22) we now compute the second derivatives:

$$\begin{aligned}
u_{xx} &= (u_{rr}r_x + u_{r\theta}\theta_x)\cos(\theta) - (u_{\theta r}r_x + u_{\theta\theta}\theta_x)\frac{\sin(\theta)}{r} \\
&\quad + u_r(\cos(\theta))_x - u_\theta\left(\frac{\sin(\theta)}{r}\right)_x \\
&= \left(u_{rr}\cos(\theta) + u_{r\theta}\left(-\frac{\sin(\theta)}{r}\right)\right)\cos(\theta) \\
&\quad - \left(u_{\theta r}\cos(\theta) + u_{\theta\theta}\left(-\frac{\sin(\theta)}{r}\right)\right)\frac{\sin(\theta)}{r} \\
&\quad + u_r(-\sin(\theta))\theta_x - u_\theta\left(\frac{(r\cos(\theta)\theta_x - \sin(\theta)r_x)}{r^2}\right) \\
&= u_{rr}\cos^2(\theta) - 2u_{r\theta}\left(\frac{\sin(\theta)\cos(\theta)}{r}\right) + u_{\theta\theta}\left(\frac{\sin^2(\theta)}{r^2}\right) \\
&\quad + u_r\frac{\sin^2(\theta)}{r} - u_\theta\left(\frac{(-\cos(\theta)\sin(\theta) - \sin(\theta)\cos(\theta))}{r^2}\right).
\end{aligned}$$

Finally we arrive at

$$\begin{aligned}
u_{xx} &= u_{rr}\cos^2(\theta) - 2u_{r\theta}\left(\frac{\sin(\theta)\cos(\theta)}{r}\right) + u_{\theta\theta}\left(\frac{\sin^2(\theta)}{r^2}\right) \\
&\quad + u_r\frac{\sin^2(\theta)}{r} + u_\theta\left(\frac{(2\sin(\theta)\cos(\theta))}{r^2}\right)
\end{aligned} \tag{23}$$

Exactly the same type of calculation which begin with

$$u_{yy} = u_r r_y + u_\theta\theta_y = u_r\sin(\theta) + u_\theta\left(\frac{\cos(\theta)}{r}\right)$$

leads to

$$\begin{aligned}
u_{yy} &= u_{rr}\sin^2(\theta) + 2u_{r\theta}\left(\frac{\sin(\theta)\cos(\theta)}{r}\right) + u_{\theta\theta}\left(\frac{\cos^2(\theta)}{r^2}\right) \\
&\quad + u_r\frac{\sin^2(\theta)}{r} - u_\theta\left(\frac{(2\sin(\theta)\cos(\theta))}{r^2}\right)
\end{aligned} \tag{24}$$

Now adding (23) and (24) leads to

$$u_{xx} + u_{yy} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}. \tag{25}$$

With this we can rewrite the problem (12) as

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad 0 < r < R, \ -\pi \le \theta \le \pi \tag{26}$$

$$u(R,\theta) = f(\theta),,$$

$$u(r,\theta) \text{ bounded.} \tag{27}$$

Separation of variables proceeds as follows. We seek simple solutions to (29) in the form

$$u = \Theta(\theta)R(r).$$

Substituting this into the equation in (29) we have

$$\left(\Theta(\theta)R(r)\right)_{rr} + \frac{1}{r}\left(\Theta(\theta)R(r)\right)_{r} + \frac{1}{r^2}\left(\Theta(\theta)R(r)\right)_{\theta\theta} = 0$$

$$\Theta(\theta)R''(r) + \frac{1}{r}\Theta(\theta)R'(r) + \frac{1}{r^2}\Theta''(\theta)R(r) = 0.$$

Next we divide by $\Theta(\theta)R(r)$ and multiply by r^2 to obtain

$$\frac{r^2(R'' + (1/r)R'}{R} = -\frac{\Theta''}{\Theta}$$

which as usual in separation of variables must equal a constant.

Recall the solutions for so-called Euler-Cauchy equations:

$$r^2R'' + arR' + bR = 0$$

Consider the change of variables $s = ln(r)$ or $r = e^s$. By the chain rule

$$\frac{dR}{dr} = \frac{dR}{ds}\frac{ds}{dr} = \frac{1}{r}\frac{dR}{ds}$$

and

$$\frac{d^2R}{dr^2} = \frac{d}{ds}\left(\frac{dR}{dr}\frac{1}{r}\right) = \frac{1}{r^2}\frac{dR^2}{ds^2} - \frac{1}{r^2}\frac{dR}{ds}$$

So the equation becomes

$$r^2\left(\frac{1}{r^2}\frac{dR^2}{ds^2} - \frac{1}{r^2}\frac{dR}{ds}\right) + ar\frac{1}{r}\frac{dR}{ds} + bR = 0$$

which simplifies to

$$\frac{dR^2}{ds^2} + (a-1)\frac{dR}{ds} + bR = 0.$$

This is a constant coefficient equation

Thus we obtain the pair of ODEs:

$$\Theta'' + \lambda\Theta = 0$$
$$\Theta(-\pi) = \Theta(\pi)$$
$$\Theta'(-\pi) = \Theta'(\pi)$$

For $\lambda = 0$ we have

$$\Theta(\theta) = 1.$$

With $\lambda = \mu^2$ the general solution is

$$\Theta(\theta) = A\cos(\mu\theta) + B\sin(\mu\theta)$$

The first boundary condition implies

$$B\sin(\mu\pi) = B\sin(-\mu\pi)$$

$$\Rightarrow \quad \sin(\mu\pi) = 0.$$

The second boundary condition implies

$$-\mu A\sin(\mu\pi) = -\mu A\sin(-\mu\pi)$$

which again gives $\lambda = \mu^2$ and both A and B are arbitrary. Thus we have $\lambda = m^2$ and

$$\Theta(\theta) = A_m\cos(m\theta) + B_m\sin(m\theta)$$

$$r^2R'' + rR' - m^2R = 0$$

which by the above discussion leads to

$$\frac{dR^2}{ds^2} + m^2R = 0.$$

When $m = 0$ we get

$$\frac{dR^2}{ds^2}$$

which implies

$$R = a + bs = a + b\ln(r).$$

But if the solution is to be bounded then we need $b = 0$ so R is an arbitrary constant, say $R = 1$.

For $m \neq 0$ we have

$$R = ae^{ms} + be^{-ms} = ar^m + br^{-m}.$$

Once again, for R bounded we need $b = 0$, so we take

$$R = r^m.$$

Combining these results we seek our general solution in the form

$$u(r,\theta) = a_0 + \sum_{m=1}^{\infty} r^m\left[a_m\cos(m\theta) + b_m\sin(m\theta)\right] \tag{28}$$

Now we need

$$f(\theta) = u(R,\theta) = a_0 + \sum_{m=1}^{\infty} R^m\left[a_m\cos(m\theta) + b_m\sin(m\theta)\right].$$

This is a general Fourier series and we have

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta$$

$$a_m = \frac{1}{\pi R^m}\int_{-\pi}^{\pi} f(\theta)\cos(m\pi\theta)\,d\theta$$

$$b_m = \frac{1}{\pi R^m}\int_{-\pi}^{\pi} f(\theta)\sin(m\pi\theta)\,d\theta$$

Example 2.

$$\begin{aligned} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} &= 0, \quad 0 < r < 1, \ -\pi \le \theta \le \pi \\ u(R,\theta) &= \cos^2(\theta), \\ u(r,\theta) &\text{ bounded.} \end{aligned}$$

Using the trig identities

$$\cos^2(\theta) = \frac{1}{2} + \frac{1}{2}\cos(2\theta)$$

so the solution is given by

$$u(r\theta) = \frac{1}{2} + \frac{1}{2}r^2\cos(2\theta)$$

Example 3 (ellipweb3).

$$\begin{aligned} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} &= 0, \quad 0 < r < 1, \ -\pi \le \theta \le \pi \\ u(R,\theta) &= \sin(3\theta), \\ u(r,\theta) &\text{ bounded.} \end{aligned}$$

$$u(r,\theta) = a_0 + \sum_{m=1}^{\infty} r^m \left[a_m\cos(m\theta) + b_m\sin(m\theta)\right]$$

Now we need

$$\sin(3\theta) = u(1,\theta) = a_0 + \sum_{m=1}^{\infty} \left[a_m\cos(m\theta) + b_m\sin(m\theta)\right].$$

But we can argue (using our knowledge of orthogonality) that the solution is given by

$$u(r,\theta) = r^3\sin(3\theta).$$

The Neumann Problem and Solvability Conditions

While the following situation holds for any Neumann problem for Laplace's equation, we choose to illustrate it with an example for the Disk.

$$\begin{aligned} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} &= 0, \quad 0 < r < R, \ -\pi \le \theta \le \pi && (29) \\ \frac{\partial u(R,\theta)}{\partial r} &= f(\theta), \\ u(r,\theta) &\text{ bounded.} && (30) \end{aligned}$$

Here $\dfrac{\partial u(R,\theta)}{\partial r}$ is the radial derivative which is also the exterior normal derivative. Thus we consider a problem in which the normal derivative is given on the boundary (Neumann

Boundary conditions) instead of the value of u itself, i.e., Diricihlet Boundary conditions. But in this case there is something new that arises.

If we view this as a problem in steady state heat flow then in order that the solution truly be steady state, i.e., does not depend on time t, we need that the flux of heat across the boundary should be zero. If it were not then then according to the laws of thermodynamics there would have to be a change in the temperature distribution with respect to time – namely, heat would have to flow from one spot to another to accommodate the nonzero flux across the boundary.

More generally, for the Neumann problem to make sense as a steady state problem we need the net gain in heat across the boundary C of a domain Ω to be zero (so there is no heat flow). That is, if $\dfrac{\partial u}{\partial n}$ denotes the interior normal derivative, then we need the total heat flux across the boundary

$$\oint_C \frac{\partial u}{\partial n} = 0$$

or else the problem has no solution.

For example the interior Neumann problem

$$\begin{aligned} \Delta u = 0, &\quad 0 < r < 1, \quad -\pi \le \theta \le \pi \\ \frac{\partial u(R,\theta)}{\partial r}(1,\theta) = 1, &\quad -\pi \le \theta \le \pi \end{aligned}$$

has no physical meaning since the constant inward flux would not give rise to a steady-state solution.

Also the Neumann problem is different in that solutions are not unique. For example, the interior Neumann problem

$$\begin{aligned} \Delta u = 0, &\quad 0 < r < 1, \quad -\pi \le \theta \le \pi \\ \frac{\partial u(R,\theta)}{\partial r}(1,\theta) = \cos(2\theta), &\quad -\pi \le \theta \le \pi \end{aligned}$$

is solvable since

$$\int_{-\pi}^{\pi} \frac{\partial u}{\partial r}(1,\theta)\, d\theta = \int_{-\pi}^{\pi} \cos(2\theta)\, d\theta = \left.\frac{\sin(2\theta)}{2}\right|_{-\pi}^{\pi} = 0.$$

But the solution is not unique since we can add any constant, i.e., every function

$$u(r,\theta) = r^2\cos(2\theta) + C, \quad C \in \mathbb{R}$$

solves the problem.

Thus to obtain a unique solution to the Neumann problem we need some additional information such as the value of the temperature at a point.

The Principle of Superposition

We note that for a linear problem it is always possible to replace a single hard problem by several simpler problems. We illustrate this idea with a simple example. Consider the problem

$$\begin{aligned} u_{xx}(x,y)+u_{yy}(x,y)&=0, \quad (x,y)\in[0,a]\times[0,b], \\ u(0,y)=f(y), &\quad u(a,y)=h(y) \\ u(x,0)=k(x), &\quad u(x,b)=f(x) \end{aligned} \tag{31}$$

The solution u can be obtained as a sum of the solutions to four simpler problems

$$\begin{aligned} u_{1,xx}(x,y)+u_{1,yy}(x,y)&=0, \quad (x,y)\in[0,a]\times[0,b], \\ u_1(0,y)=f(y), &\quad u_1(a,y)=0 \\ u_1(x,0)=0, &\quad u_1(x,b)=0 \end{aligned} \tag{32}$$

$$\begin{aligned} u_{2,xx}(x,y)+u_{2,yy}(x,y)&=0, \quad (x,y)\in[0,a]\times[0,b], \\ u_2(0,y)=0, &\quad u_2(a,y)=0 \\ u_2(x,0)=, &\quad u_2(x,b)=g(x) \end{aligned} \tag{33}$$

$$\begin{aligned} u_{3,xx}(x,y)+u_{3,yy}(x,y)&=0, \quad (x,y)\in[0,a]\times[0,b], \\ u_3(0,y)=0, &\quad u_3(a,y)=h(y) \\ u_3(x,0)=0, &\quad u_3(x,b)=0 \end{aligned} \tag{34}$$

$$\begin{aligned} u_{4,xx}(x,y)+u_{4,yy}(x,y)&=0, \quad (x,y)\in[0,a]\times[0,b], \\ u_4(0,y)=0, &\quad u_4(a,y)=0 \\ u_4(x,0)=k(x), &\quad u_4(x,b)=0 \end{aligned} \tag{35}$$