Thm Spectrum 1 For $T \in B(X)\), $X$ a Banach space define $\Phi_T = \{ \lambda \in \mathbb{C} : (\lambda I - T) \in \Phi(X) \}$. If $K \in K(X), \lambda \neq 0, A = \lambda I - K$ then $\Phi_A = \mathbb{C} \setminus \{ 0 \}$.

**Definition** Let $A \in B(X)$

(a) If $\lambda \in \mathbb{C}$ and $\dim(N(\lambda - A)) > 0$, then $\lambda$ is called an *eigenvalue* of $A$ and a nonzero vector $x$ such that $(\lambda - A)x = 0$ is called an *eigenvector*.

(b) $\rho(A) = \{ \lambda : (\lambda - A) \text{ has a bounded inverse } \}$ is called the *resolvent set* of $A$.

(c) $\sigma(A) = \mathbb{C} \setminus \rho(A)$ is called the *spectrum* of $A$. The spectrum of $A$ is further partitioned into its *point spectrum* denoted by $\sigma_p(A)$, *continuous spectrum* denoted by $\sigma_c(A)$, and *residual spectrum* denoted by $\sigma_r(A)$:

(i) $\sigma_p(A) = \{ \lambda \in \mathbb{C} : \dim(N(\lambda - A)) > 0 \} = \{ \text{eigenvalues of } A \}$

(ii) $\sigma_c(A) = \{ \lambda \in \mathbb{C} : N(\lambda - A) = \{ 0 \}, \overline{R(\lambda - A)} = X, (\lambda - A)^{-1} \notin B(X) \}$

(iii) $\sigma_r(A) = \{ \lambda \in \mathbb{C} : N(\lambda - A) = \{ 0 \}, \overline{R(\lambda - A)} \text{ not dense in } X \}$

**Thm Spectrum 2** $X$ a Banach space $K \in K(X), A = (\lambda I - K)$,

(a) then $\dim(N(A)) = 0$ except for an (at most countable) set $S = \{ \lambda : \dim(N(A)) > 0 \}$. 

---
The set $S$ depends on $K$ and $0$ is the only possible limit point.

If $\lambda \neq 0$ and $\lambda \not\in S$, then $\dim(N(A)) = 0$, $R(\lambda - K) = X$ and $(\lambda - K)$ is invertible.

**Note** We have shown in **Thm Spectrum 2** that for $K \in K(X)$

(a) $\sigma_p(K)$ is an at most countable set $S$ which has $0$ as the only possible limit point.

(b) All points $\lambda \neq 0, \lambda \not\in S = \{ \lambda : \dim(N(\lambda - K)) > 0 \}$ are in $\rho(K)$.

**Thm Spectrum 3** $\rho(A)$ is open $\Rightarrow \sigma(A)$ is closed. For $\lambda \in \rho(A)$ the resolvent operator $R(\lambda) = (\lambda - A)^{-1}$ for $\lambda_0 \in \rho(A)$ and $|\lambda - \lambda_0| < 1/\|R(\lambda_0)\|$ satisfies

$$R(\lambda) = R(\lambda_0)[I + (\lambda - \lambda_0)R(\lambda_0)]^{-1} = \sum_{k=0}^{\infty} (-1)^k R(\lambda_0)^{k+1} (\lambda - \lambda_0)^k$$

where the last term is a convergent series in $X$.

**Thm Spectrum 4** For $A \in B(X)$, define the spectral radius $r_\sigma(A)$ by

$$r_\sigma(A) = \inf_n \|A^n\|^{1/n}.$$ 

Then

$$\{ \lambda : |\lambda| > r_\sigma(A) \} \subset \rho(A).$$

**Lemma Spectrum 5** If $|\lambda| > \|A\|$, $\Rightarrow \lambda \in \rho(A)$ and if $\lambda \in \sigma(A)$ then $\lambda^n \in \sigma(A^n)$.

**Lemma Spectrum 6** If $M$, and $K$ are two operators such that $MK$ is invertible then $K$ is one-to-one, and $M$ is onto. So if $M$ and $K$ commute, i.e., $MK = KM$ is invertible then both $M$ and $K$ are one-to-one and onto and therefore they are both invertible.

**Thm Spectrum 7** Let $p(t) = \sum_{j=1}^{n} a_k t^k$ be a polynomial and define $p(A) = \sum_{j=0}^{n} a_k A^k$ where $A^0 = I$. Then $\lambda \in \sigma(A)$ implies $p(\lambda) \in \sigma(p(A))$. If $X$ is a complex Banach space then $p(\sigma(A)) = \sigma(p(A))$.

**Corollary Spectrum 8** If $X$ is a complex Banach space, then the equation $p(A)x = y$ is solvable for any $y \in Y \iff p(\lambda) \neq 0$ for all $\lambda \in \sigma(A)$.

**Theorem Spectrum 9** $R(\lambda) = (\lambda - A)^{-1}$ is analytic on $\pi(A)$.
(\lambda - A) is one-to-one

Yes

R(\lambda - A) is dense

Yes

(\lambda - A)^{-1}
bounded on R(\lambda - A)

Yes

\lambda \in \rho(A)

No

\lambda \in \sigma_p(A)

Yes

\lambda \in \sigma_r(A)

No

\lambda \in \sigma_c(A)
Lemma Spectrum 10 If $|z| > \limsup_n \|A^n\|^{1/n}$ then $(z - A)^{-1} = \sum_{j=1}^{\infty} z^{-n}A^{n-1}$ converges in $B(x)$.

Thm Spectrum 11 If $\sigma(A) \subset \Omega$ an open set in $\mathbb{C}$ whose boundary $C = \partial \Omega$ is a Simple Closed (Positively) Oriented Rectifiable Curve (SCROC) then

$$A^n = \frac{1}{2\pi i} \oint_C z^n R(z) \, dz.$$  

Lemma Spectrum 12 $r_\sigma(A) = \max_{\lambda \in \sigma(A)} |\lambda|$ and $\lim_{n \to \infty} \|A^n\|^{1/n} = r_\sigma(A)$.

Thm Spectrum 13 If $C$ is a circle with radius $R$ where $R > r_\sigma(A)$ and $f \in \mathcal{H}(\Omega)$ with $f(z) = \sum a_j z^j$ then

$$f(A) = \frac{1}{2\pi i} \oint_C f(z) R(z) \, dz.$$  

Thm Spectrum 14 If $\sigma(A) \subset \Omega$ an open set in $\mathbb{C}$ whose boundary $C = \partial \Omega$ is a SCROC and $f, g \in \mathcal{H}(\Omega)$ so that $h = fg \in \mathcal{H}(\Omega)$ then we have $h(A) = f(A)g(A)$ and

$$h(A) = f(A)g(A) = \frac{1}{2\pi i} \oint_C f(z)g(z) R(z) \, dz.$$  

Thm Spectrum 15 (The Resolvent Identity) If $\lambda, \mu \in \rho(A)$ then

$$R(\lambda) - R(\mu) = (\mu - \lambda)R(\lambda)R(\mu)$$

and we can see that $R(\lambda)R(\mu) = R(\mu)R(\lambda)$. Furthermore, for $|\lambda - \mu||R(\mu)|| < 1$ implies

$$(\lambda - A)^{-1} = \sum_{j=1}^{\infty} (\mu - \lambda)^{n-1} R(\mu)^n.$$  

Thm Spectrum 16 If $A \in B(X), f \in \mathcal{H}(\Omega)$ and $\sigma(A) \subset \Omega$ with $f(z) \neq 0$ on $\sigma(A)$, then $f(A)^{-1}$ exists and is given by

$$f(A)^{-1} = \frac{1}{2\pi i} \oint_C \frac{1}{f(z)} R(z) \, dz.$$  

Thm Spectrum 17 If $A \in B(X), f \in \mathcal{H}(\Omega)$ and $\sigma(A) \subset \Omega$, then $\sigma(f(A)) = f(\sigma(A))$.

Example Spectrum 18 Let $X$ be a complex, separable Hilbert space, $\{x_j\}$ an orthonormal basis. Thus for every $x \in X$ we have $x = \sum_{j=1}^{\infty} \alpha_j x_j$, with $\alpha_j = \langle x, x_j \rangle$. Let $\{\lambda_j\}$ satisfy $\lambda_j \uparrow 1$, $\lambda_j \neq 1$ for all $j$ and $1$ is the only limit point. Then define $Ax = \sum_{j=1}^{\infty} \alpha_j \lambda_j x_j$. Then the point spectrum $\sigma_p(A)$ consists of the numbers $\lambda_j$. The continuous spectrum is the number $1$. The remainder of the complex plane is in the resolvent set, $\rho(A)$. The residual spectrum is empty.
Example Spectrum 19 Let $X$ be a complex, separable Hilbert space, $\{x_j\}$ an orthonormal basis. Thus for every $x \in X$ we have $x = \sum_{j=1}^{\infty} \alpha_j x_j$, with $\alpha_j = \langle x, x_j \rangle$. Then define $Ax = \sum_{j=1}^{\infty} \frac{\alpha_j}{(j+1)} x_{j+1}$. $0$ is in the residual spectrum $\sigma_r(A)$. The remainder of the complex plane is in the resolvent set, $\rho(A)$.

Thm Spectrum 20 A Banach space $X$ is reflexive $\iff X'$ is reflexive.

Thm Spectrum 21 A closed subspace of a reflexive space is reflexive.

Thm Spectrum 22 A uniformly convex Banach space is reflexive.

Thm Spectrum 23 (Eberlein-Shmulyan Theorem) A Banach space is reflexive $\iff$ every bounded sequence in $X$ has a weakly convergent subsequence.

Thm Spectrum 24 $L^p$, $\ell^p$ for $1 < p < \infty$ are reflexive. All Hilbert spaces are reflexive.

Thm Spectrum 25 Spaces that are not reflexive include $c_0$, $C(a,b)$, $\ell^1$, $\ell^\infty$, $L^1$, $L^\infty$.

Thm Spectrum 26 If $T \in B(X,Y)$, then $T'' : X'' \to Y''$ is an extension of $T$. If $X$ is reflexive then $T'' = T$.

Thm Spectrum 27 The mapping $T \to T'$ is an isometric isomorphism of $B(X,Y)$ into $B(Y',X')$. If $T \in B(X,Y), U \in B(Y,Z)$. Then $(UT)' = T'U'$. The adjoint of the identity in $B(X)$ is the identity in $B(X')$.

Thm Spectrum 28 A linear operator $T \in B(X,Y)$ has a bounded inverse $T^{-1}$ defined on all of $Y$ $\iff T'$ has a bounded inverse $(T')^{-1}$ defined on all of $X'$. When the exist we have $(T^{-1})' = (T')^{-1}$.

Thm Spectrum 29 Let $T \in B(X)$, $T' \in B(X')$, then $\sigma(T') = \sigma(T)$.

Thm Spectrum 30 Let $H = L^2(\mathbb{R})$ and consider the subspace $D(A) = \{ \varphi \in H : \varphi \in AC(0,1) \text{ and } \varphi' \in H \}$. With this domain we define the linear unbounded operator $A = d/dx$. We showed the following:

(a) For all $\lambda \in \mathbb{C}$, $(\lambda - A)$ is one-to-one.

(b) The resolvent set is given by $\rho(A) = \{ \lambda \in \mathbb{C} : \Re(\lambda) \neq 0 \}$.

(c) The continuous spectrum is $\sigma_c(A) = \{ \lambda \in \mathbb{C} : \Re(\lambda) = 0 \}$.

(d) The residual spectrum is empty.
Thm Spectrum 31 Let \( H = L^2(\mathbb{R}_+) \) (where \( \mathbb{R}_+ = [0, \infty) \)) and consider the subspace \( D(A) = \{ \varphi \in H : \varphi \in AC(0, 1) \text{ and } \varphi' \in H, \varphi(0) = 0 \} \). With this domain we define the linear unbounded operator \( A = d/dx \). We showed the following:

(a) For all \( \lambda \in \mathbb{C} \), \( (\lambda - A) \) is one-to-one.

(b) The resolvent set is given by \( \rho(A) = \{ \lambda \in \mathbb{C} : \Re(\lambda) < 0 \} \).

(c) The residual spectrum is given by \( \sigma_r(A) = \{ \lambda \in \mathbb{C} : \Re(\lambda) > 0 \} \).

(d) The continuous spectrum is \( \sigma_c(A) = \{ \lambda \in \mathbb{C} : \Re(\lambda) = 0 \} \).

Thm Spectrum 32 Let \( H = L^2(\mathbb{R}_+) \) (where \( \mathbb{R}_+ = [0, \infty) \)) and consider the subspace \( D(A) = \{ \varphi \in H : \varphi \in AC(0, 1) \text{ and } \varphi' \in H \} \). With this domain we define the linear unbounded operator \( A = d/dx \). We showed the following:

(a) The point spectrum is given by \( \sigma_p(A) = \{ \lambda \in \mathbb{C} : \Re(\lambda) < 0 \} \).

(b) The resolvent set is given by \( \rho(A) = \{ \lambda \in \mathbb{C} : \Re(\lambda) > 0 \} \).

(c) The continuous spectrum is \( \sigma_c(A) = \{ \lambda \in \mathbb{C} : \Re(\lambda) = 0 \} \).

Res. Id. 1 A sequence of operators \( \{P_k\} \) on a Hilbert space \( H \) is called a Resolution of the Identity if

(a) each \( P_k \) is an orthogonal projection, (b) \( I_H = \sum P_k \). (Here \( I_H \) is the identity on \( H \) and \( \sum \) means the convergence of the sum is considered in the strong operator topology).

Theorem Res. Id. 2 From Naylor and Sell, [5].

(a) Let \( \{P_k\} \) be a Resolution of the Identity on Hilbert space \( H \) and \( R(P_k) \) be the range of \( P_k \). Then \( R(P_k) \perp R(P_k) \) for all \( j \neq k \) and \( H = \sum R(P_k) \).

(b) Conversely, let \( \{R_k\} \) be a collection of closed linear subspaces of a Hilbert space \( H \) with \( R_j \perp R_k \) for \( j \neq k \) and such that \( H = \sum R_k \). Let \( P_k \) be thr orthogonal projection onto \( R_k \). Then \( \{P_k\} \) is a Resolution of the Identity.

Res. Id. 3 Let \( H \) be a Hilbert space and let \( \{P_k\} \) be a Resolution of the Identity on \( H \) and let \( \{\lambda_k\} \) be a sequence of scalars. A linear map of the form

\[
T \varphi = \sum_{k=1}^{\infty} \lambda_k P_k \varphi \quad \text{for} \quad \varphi \in D(T)
\]
where
\[ D(T) = \{ \varphi \in H : \lim_{N \to \infty} \sum_{k=1}^{N} \lambda_k P_k \varphi \text{ exists} \}. \]
is called a weighted sum of projections.

**Theorem Res. Id. 3** Let \( T \) be a weighted sum of projections, then we have the following results:

(a) \( D(T) = H \) if and only if \( \{ |\lambda_k| \} \) are bounded.

(b) For any set \( \{ |\lambda_k| \} \), \( \overline{D(T)} = H \).

(c) \( T \) is continuous if and only if \( \{ |\lambda_k| \} \) are bounded if and only if \( D(T) = H \). In this case \( \|T\| = \sup \{ \lambda_k \} \).

(d) \( (\lambda - T) \) is one-to-one if and only if \( \lambda \neq \lambda_k \) for all \( k \).

(e) \( \overline{R(\lambda - T)} = H \) if, and only if \( \lambda \neq \lambda_k \) for all \( k \).

(f) \( R(\lambda - T) = H \) if, and only if \( \{ |\lambda - \lambda_k| \} \) are bounded away from zero.

(g) \( \overline{D(T)} = H \) for every sequence \( \{ \lambda_k \} \).

(h) If \( \lambda \neq \lambda_k \) for all \( k \), then \( (\lambda - T) \) has a bounded inverse on its range given by

\[ (\lambda - T)^{-1} \psi = \sum_k (\lambda - \lambda_k)^{-1} P_k \psi \]

for all \( \psi \in R(\lambda - T) \). The operator \( (\lambda - T)^{-1} \) is bounded if, and only if \( \{ |\lambda - \lambda_k| \} \) are bounded away from zero.

(i) If \( T = \sum \lambda_k P_k \) where \( \{ P_k \} \) is a Resolution of the Identity and if \( T \) is bounded then the adjoint \( T^* \) is given by \( T^* = \sum \lambda_k P_k \).

(j) A weighted sum of projections is self-adjoint if, and only if the \( \{ \lambda_k \} \) are real. But every bounded weighted sum of projections defines a normal operator.

(k) A weighted sum of projections is compact if i. for every nonzero \( \lambda_k \) the range of \( P_k \) is finite dimensional, ii. for every \( \epsilon > 0 \) the number of \( \lambda_k \)'s such that \( |\lambda_k| > \epsilon \) is finite.

**Spectral Properties 1** We reiterate what was learned in **Thm Spectrum 2** (see also the note immediately thereafter). If \( T \) is compact in a Hilbert space the every \( \lambda \in \mathbb{C} \) is either an eigenvalue with finite multiplicity or it is in the resolvent set. (That is, the continuous and residual spectrums are empty). We also showed last semester that the range \( R(\lambda - T) \) is closed for all \( \lambda \neq 0 \). For every
\( \epsilon > 0 \), the number of eigenvalues in modulus greater than \( \epsilon \) is finite. The spectrum of \( T \) is at most countable and \( \lambda = 0 \) is the only possible accumulation point.

**Spectral Properties 2** If \( T \) is a normal operator on a Hilbert space \( H \)

(a) If \( \lambda, \varphi \) an eigenpair for \( T \), then \( \varphi \) is a eigenvector for \( T^* \) with eigenvalue \( \overline{\lambda} \), and

\[
N(\lambda - T) = N(\overline{\lambda} - T^*).
\]

(b) If \( \lambda \neq \mu \) are eigenvalues of \( T \) then \( N(\lambda - T) \perp N(\overline{\lambda} - T^*) \).

(c) For each \( \lambda \in \mathbb{C} \), the close subspace \( N(\lambda - T) \) reduces \( T \). We say that a subspace \( M \) reduces \( T \) if \( TM \subset M \) and \( TM^\perp \subset M^\perp \).

(d) If \( \{M_j\} \) is a family of eigen-subspaces of \( T \), then \( M = \sum M_j \) reduces \( T \).

(e) The residual spectrum of \( T \) is empty.

(f) A complex number \( \lambda \) is in \( \sigma(T) \) if, and only if there exists a sequence, \( \{\varphi_k\} \), \( \|\varphi_k\| = 1 \) for all \( k \) such that \( \|(\lambda - T)\varphi_k\| \to 0 \) as \( k \to \infty \), i.e., \( (\lambda - T) \) is not bounded below.

(g) \( T \) can be written uniquely in the form \( T = A + iB \) where \( A, B \) are self-adjoint. Furthermore, \( T^* = A - iB \). Also we have
\[
\max(\|A\|, \|B\|) \leq \|T\| = \|T^*\| \leq \|A\|^2 + \|B\|^2.
\]

(h) \( T = A + iB \) (as above) is compact if, and only if \( A \) and \( B \) are compact.

**Spectral Properties 3** Let \( T \) be a bounded self-adjoint operator of a Hilbert space \( H \)

(a) The spectrum \( \sigma(T) \) is a subset of the real interval \([-\|T\|, \|T\|]\).

(b) If \( T \) is compact and self-adjoint then \( T \) has an eigenvalue \( \lambda \) with \( |\lambda| = \|T\| \).

**Spectral Theorem 1** (First Version) Let \( T \) be compact normal on \( H \). Then there is a Resolution of the Identity \( \{P_k\} \) and a sequence of complex numbers \( \{\lambda_k\} \) such that
\[
T = \sum_{k} \lambda_k P_k
\]

(note the convergence is in the strong operator topology, i.e., this is a weighted sum of projections).
Spectral Theorem 2 (Second Version) Let $T$ be compact normal on $H$. Then there is an orthonormal basis of eigenvectors $\{e_k\}$ and corresponding eigenvalues $\{\mu_k\}$ such that if $x = \sum_n \langle x, e_k \rangle e_k$ is the Fourier expansion of $x$, then

$$Tx = \sum_n \mu_k \langle x, e_k \rangle e_k.$$  

Func Calc 1 Let $T$ be compact normal on Hilbert space $H$. Then $T$ can be expressed as a weighted sum of projections

$$T = \sum_n \lambda_n P_n, \quad T^* = \sum_n \overline{\lambda_n} P_n$$

by the spectral theorem. We set $T^0 = I$.

(a) If $p(z) = \sum_{k=0}^n \alpha_k z^k$ then $p(T) = \sum_{k=0}^n \alpha_k P^k$.

(b) If $p(z) = \sum_{k=0}^n \alpha_k z^k$ then $p(T) = \sum_{k=0}^n \alpha_k T^k$.

(c) If $p(z, \overline{z}) = \sum_{j,k=0}^n \alpha_{j,k} z^j \overline{z}^k$ then $p(T, T^*) = \sum_{j,k=0}^n \alpha_{j,k} P^j (T^*)^k$ and we have

$$p(T, T^*) = \sum_n p(\lambda_n, \overline{\lambda_n}) P_n.$$  

(d) If $p(z)$ and $q(z)$ are two polynomials, where $q(z)$ has not zeros on the spectrum $\sigma(T)$, and $r(z) = p(z)q(z)^{-1}$, then

$$p(T)q(T)^{-1} = \sum_n \lambda_n P_n.$$  

(e) If $p(z, \overline{z})$ and $q(z, \overline{z})$ are two polynomials in $z$ and $\overline{z}$, where $q(z, \overline{z})$ has not zeros on the spectrum $\sigma(T) \times \sigma(T^*)$, and $r = pq^{-1}$, then

$$r(T^*, T) = \sum_n r(\lambda_n, \overline{\lambda_n}) P_n.$$  

(f) If $f(z)$ is a continuous function defined on the spectrum $\sigma(T)$, then

$$f(T) = \sum_n f(\lambda_n) P_n.$$  

Unbdd 1 (a) Let $\mathcal{L}(X,Y)$ denote the set of all linear operators mapping from $X$ to $Y$. Here $X$ and $Y$ are normed (or Banach) spaces. $A \in \mathcal{L}(X,Y)$ is a only assumed to be defined on a subspace $D(A) \subset X$ called the domain of $A$. 

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(b) As an example, if $X = Y = C[0, 1]$, $D(A) = C^1[0, 1]$ and $A = d/dx$. Then $A \in \mathcal{L}(X, Y)$ and it is easy to see that $A$ is not bounded (i.e., not continuous). Recall the norm of $\varphi \in C[0, 1]$ is $\|\varphi\|_{\infty} = \sup_{x \in [0, 1]} |\varphi(x)|$. Clearly $\overline{D(A)} = C[0, 1]$ so $A$ is densely defined and $\varphi_n = x^n \in D(A)$ with $\|\varphi_n\| = 1$ for all $n$ but $\|A\varphi_n\| = \|nx^{n-1}\| = n \to \infty$ as $n \to \infty$.

(c) $G(A) = \{\{x, y\} \in X \times Y : x \in D(A), y = Ax\}$ is called the Graph of $A$.

(d) A subspace $V \subset X \times Y$ is the graph of a linear map if, and only if $\{0, y\} \not\subset V$ for all $y \neq 0$. If this condition holds then we can define $A$ as follows: the domain of $A$ is $D(A) = \{x \in X : \{x, y\} \in V\}$ and for $x \in D(A)$ we define $Ax = y$.

(e) $A \in \mathcal{L}(X, Y)$ is said to be one-to-one (or injective) if

iff $N(A) = \{0\}$ (the null space of $A$).

(f) If $A$ is one-to-one we define $A^{-1} : Y \to X$ by $D(A^{-1}) = R(A)$ (the range of $A$) and $A^{-1}y = x$ if, and only if $y = Ax$. Thus the $G(A^{-1}) = \{\{y, x\} : \{x, y\} \in G(A)\}$.

Note that if $A$ is not one-to-one then there exists an $x \neq 0$ such that $Ax = 0$ which implies $\{x, 0\} \in G(A)$ which implies that $\{0, x\} \in G(A^{-1})$ which is not the graph of a function.

**Unbdd 2** If $A : D(A) \subset X \to Y$, $A \in \mathcal{L}(X, Y)$ then we say $A$ is closed if for every $\{x_n\} \subset D(A)$ such that $x_n \to y$ and $Ax_n \to y$ then $x \in D(A)$ and $Ax = y$.

**Unbdd 3** The following are equivalent:

(a) $A$ is closed

(b) $G(A)$ is closed in $X \times Y$

(c) $D(A)$ is complete in the graph norm, $\|x\|_{D(A)} = \|x\|_X + \|Ax\|_Y$.

The example of $A = d/dx$ given in **Unbdd 1** is closed (this was a homework assignment last semester).

**Unbdd 4** If $A : D(A) \subset X \to Y$, $A \in \mathcal{L}(X, Y)$ then we say $A$ is closable if for every $\{x_n\} \subset D(A)$ such that $x_n \to 0$ and $Ax_n \to y$ then $y = 0$. The is the same as saying that $\overline{G(A)}$ is a graph.

**Unbdd 5** When $A$ is closable there exists a closed operator $\overline{A}$ with $G(\overline{A}) = \overline{G(A)}$. $\overline{A}$ is the smallest closed extension of $A$, i.e., if $B$ is closed and is an extension of $A$, then $B$ is also an extension of $\overline{A}$.

**Unbdd 6** Let $X = Y = C[0, 1]$ and $D(A_0) = C^\infty[0, 1]$ with $A_0 = d/dx$. Then $A_0$ is not closed but it is closable and $\overline{A}_0$ is the operator $A$ given in **Unbdd 1**.

**Unbdd 7** Another class of examples operators that are not closed can be obtained as follows: Take any subspace $V \subset X$ which is NOT closed and define $A : D(A) = V \subset X \to X$ by $Ax = x$. Note
that \( A \) is bounded but not closed. Since if \( x \in V \setminus \overline{V} \) then there exists \( \{x_n\} \subset V \) and \( x \in X \) such that \( x_n \rightarrow x \) and \( Ax_n = x_n \rightarrow x \) but \( x \in D(A) \) so \( A \) is not closed.

Note that a bounded linear map \( A \) on a closed domain is closed but there are many unbounded closed operators.

**Unbdd 8** For any \( A \in \mathcal{L}(X,Y) \) defined on the domain \( D(A) \subset X \), there are many adjoints mapping \( Y' \) to \( X' \). However, if \( A \) is densely defined there is a unique maximal operator denoted \( A' \) which is adjoint to \( A \), i.e.,

\[
y'(Ax) = (A'y)(x) \quad \forall \ x \in D(A), \quad y \in D(A').
\]

By maximal we mean that if \( S \) satisfies the above with \( A' = S \), then \( G(S) \subset G(A') \).

The unique maximal adjoint is given as follows

\[
D(A') = \{g \in Y' : \exists f \in X' \text{ such that } g(Ax) = f(x) \ \forall \ x \in D(A)\}
\]

and we then define for \( g \in D(A') \), \( A'(g) = f \). The fact that \( \overline{D(A)} = X \) implies that \( f \in X' \) is uniquely defined and we have

\[
g(Ax) = (A'g)(x) \quad \forall \ x \in D(A), \quad g \in D(A').
\]

**Unbdd 9** If \( X \) is a Banach space and \( A \in B(X) \) then the Banach space adjoint (or dual map), \( A' \in B(X') \) and we have

(a) If \( \lambda \in \sigma_r(A) \) (the residual spectrum of \( A \)) then \( \lambda \in \sigma_p(A') \) (point spectrum of \( A' \)).

(b) If \( \lambda \in \sigma_p(A) \) (the point spectrum of \( A \)) then \( \lambda \in \sigma_p(A') \) (residual spectrum of \( A' \)) or \( \lambda \in \sigma_p(A') \) (the point spectrum of \( A' \)).

**Unbdd 10** If \( H \) is a Hilbert space and \( A \) is a densely defined linear operator with domain \( D(A) \), then we define the *Hilbert space adjoint* \( A^* \) of \( A \) as follows:

\[
D(A^*) = \{y \in H : \exists x^* \in H \text{ such that } \langle Ax, y \rangle = \langle x, x^* \rangle \ \forall \ x \in D(A)\}.
\]

This is the same as the set of all \( y \in H \) such that the linear functional \( F(x) = \langle Ax, y \rangle \) defined on \( D(A) \) extends to a continuous linear functional define on all of \( X \).

Then for every \( y \in D(A^*) \) we define \( A^*y = x^* \).

With this definition we have

\[
\langle Ax, y \rangle = \langle x, A^*y \rangle \ \forall \ x \in D(A), \quad y \in D(A^*).
\]
Let $A \in \mathcal{L}(H), \overline{D(A)} = H$, then

(a) $A^*$ is closed

(b) $S$ is closable if, and only if $D(A^*)$ is dense. If $A$ is closable then $(A)^* = A^*$.

(c) If $A$ is densely defined and closed, then (see previous) $A^{**}$ exists and $A^{**} = A$.

(d) $N(A^*) = R(A)^\perp = \{x \in H : \langle x, Az \rangle = 0 \ \forall z \in D(A)\}$.

(e) $A$ closed implies $N(A) = R(A^*)^\perp$.

(f) If $R(A)$ is closed then $R(A^*)$ is closed and

$$R(A) = N(A^*)^\perp, \quad R(A^*) = N(A)^\perp.$$  

(g) If $A^{-1}, A^*$ and $(A^{-1})^*$ exist, then $(A^*)^{-1}$ exists and $(A^*)^{-1} = (A^{-1})^*$

**Unbdd 12**

**Unbdd 13** Let $H$ be an inner product space and define an inner product on $H \times H$ by

$$\langle\{a, b\}, \{c, d\}\rangle_{H \times H} = \langle a, c \rangle + \langle b, d \rangle,$$

and the induced norm is given by

$$\|\{a, b\}\|_{H \times H} = \|a\|^2 + \|b\|^2.$$

**Unbdd 14** Define $U, V : H \times H \to H \times H$ by $V\{a, b\} = \{-b, a\}$ and $U\{a, b\} = \{b, a\}$.

**Unbdd 15** $G(A^*) = [VG(A)]^\perp$.

**Unbdd 16** $VG(A^*) = [G(A)]^\perp$.

**Unbdd 17** Let $L = d/dx$ denote the differentiation operator acting in the Hilbert space $H = L^2(a, b)$ where $-\infty < a < b < \infty$.

(a) $T_{\text{max}} = L$ with

$$D(T_{\text{max}}) = \{\varphi \in AC(a, b) : \varphi' \in H\}$$

i.e., recall from real analysis, $\varphi \in AC(a, b)$ implies $\varphi = \int \psi$ with $\psi \in L^1$ and $\varphi' = \psi$ a.e. For $\varphi$ to be in the $D(T)$ we require that this $\psi \in L^2$.

(b) Define $T_1$ as the restriction of $T_{\text{max}}$ with domain

$$D(T_1) = \{\varphi \in AC(a, b) : \varphi' \in H, \ \varphi(a) = 0\}$$
(c) We define \( T_2 \) as the restriction of \( T_{\text{max}} \) with domain

\[
D(T_2) = \{ \varphi \in \text{AC}(a,b) : \varphi' \in H, \ \varphi(b) = 0 \}
\]

(d) Define \( T_3(k) \) as the restriction of \( T_{\text{max}} \) subject to the boundary conditions

\[
\varphi(b) = k\varphi(a) \ \forall \ k \in \mathbb{R}, \ \text{and} \ u(b) = 0 \ \text{for} \ k = \infty.
\]

That is, we define

\[
D(T_3(k)) = \{ \varphi \in \text{AC}(a,b) : \varphi' \in H, \ \varphi(b) = k\varphi(a) \}
\]

if \( k \in \mathbb{R} \) and for \( k = \infty \)

\[
D(T_3(k)) = \{ \varphi \in \text{AC}(a,b) : \varphi' \in H, \ \varphi(b) = 0 \}
\]

(e) Define \( T_{00} \) by

\[
D(T_{00}) = \{ \varphi \in \text{AC}(a,b) : \varphi' \in H, \ \varphi(a) = \varphi(b) = 0 \}.
\]

Note that \( D(T_{\text{min}}) \subset D(T_{00}) \).

(f) We finally define \( T_{\text{min}} \) by

\[
D(T_{\text{min}}) = C_0^\infty(a,b)
\]

(infinitely differentiable functions with compact support in \((a,b)\)). Note that \( D(T_{\text{min}}) \subset D(T_j) \) for all \( j \).

**Unbdd 18** Some results for these examples:

(a) Since \( C_0^\infty(a,b) \) is dense in \( H = L^2(a,b) \) we see that all of the above operators are densely defined.

(b) \( T_{\text{max}} \) is not invertible since it is not one-to-one. In particular \( \varphi = c \in \mathbb{R} \) (constant functions) are in the null space of \( T_{\text{max}} \).

(c) \( T_1 \) and \( T_2 \) have bounded inverses.

(d) \( T_{00} \) is invertible (but only on the range of \( T_{00} \)):

\[
R(T_{00}) = \left\{ \varphi \in H : \int_a^b \varphi(x) \, dx = 0 \right\}.
\]
(e) The operator $T_3(k)$ for $k \in \mathbb{R} \setminus k = 1$ is invertible. Note that $T_3(\infty) = T_1$ and $T_3(0) = T_2$, both of which we have already shown to be invertible.

(f) $T_{\text{max}}^* = -T_{00}$.

(g) $T_1^* = -T_2$.

(h) $T_2^* = -T_1$.

(i) $T_3(k)^* = -T_3(1/k)$ for all $k$.

(j) Note that if we had used $id/dx$ instead of $d/dx$ then $T_3(k)$ would be self-adjoint if, and only if $k = \pm 1$.

(k) Note that we have shown $T_1, T_2, T_3(k)$ for $k \neq 1$ are boundedly invertible and $T_{00}$ is boundedly invertible on its range (which is closed subspace). We also know that a bounded operator is closed and that an operator $A$ is closed if, and only if its inverse is closed. Thus we see that $T_1, T_2, T_3(k)$ for $k \neq 1$ and $T_{00}$ are all closed.

(l) We showed, as a homework assignment, that $T_{\text{max}}$ is closed.

(m) $T_{\text{min}}$ is not closed but $\overline{T_{\text{min}}} = T_{\text{max}}$.

(n) As for $T_3(1)$ we have shown that $T_3(1) = -T_3(1)^*$ and since the adjoint of a densely defined operator is always closed we see that $T_3(1)$ is also closed.

(o) $\sigma_p(T_{\text{max}}) = \mathbb{C}$, $\sigma_p(T_{\text{min}}) = \emptyset$, $\sigma_r(T_{\text{min}}) = \mathbb{C}$.

(p) For $k \neq 1$, $\sigma_p(T_3(k)) = \{-\ln |k| + i \arg(1/k) + 2\pi ij, \ j = 0, \pm 1, \cdots \}$. If $k = 1$ then $0 \notin \sigma_p(T_1)$.

Unbdd 19 An operator $A \in B(H)$ is unitary if, and only if $A^*A = AA^* = I$ if, and only if $R(A) = H$ and $\langle Ax, Ay \rangle = \langle x, y \rangle$ for all $x, y \in H$ if, and only if $R(A) = H$ and $\|Ax\| = \|x\|$ for all $x \in H$.

Unbdd 20 The operators $U$ and $V$ are unitary. In fact $V^2 = -I$ and $U^2 = I$.

Unbdd 21 Let $H$ be the Hilbert space $L^2(0,1)$. We define the linear operator $C$ as point evaluation at $x = 1$, i.e.,

$$C\varphi = \varphi(1), \ \forall \ \varphi \in D(C) = H^1(0,1).$$

Then there is a sequence $\{\varphi_n\}_{n=1}^\infty \subset H^1(0,1)$ such that

$$\varphi_n \xrightarrow{n \to \infty} 0 \ \text{and} \ C\varphi_n \xrightarrow{n \to \infty} \psi = 1$$

and therefore $C$ is not closable.
\[ \varphi_n(x) = \begin{cases} 
0 & , \quad 0 \leq x \leq 1 - \frac{1}{n} \\
(2n-1) \left( x - 1 + \frac{1}{n} \right) & , \quad 1 - \frac{1}{n} \leq x \leq 1 - \frac{1}{2n} \\
x & , \quad 1 - \frac{1}{2n} \leq x \leq 1 
\end{cases} \]

Unbdd 22 We again give some results from Naylor and Sell, [5]. These results are related to the idea of unbounded self-adjoint operators with compact resolvent. We give some definitions, results and examples. Given a linear operator \( L \) on a Separable Hilbert space \( H \). We are interested in conditions under which there exists a \( \lambda_0 \in \mathbb{C} \) in order that \( (\lambda_0 - L)^{-1} \) is self-adjoint. This situation arises very often in applications to differential equations.

(a) If \( L \) is as above with domain \( D(L) \). The operator \( L \) is said to be \textit{Symmetric} if

\[ \langle Lf, g \rangle = \langle f, Lg \rangle \quad \text{for all} \quad f, g \in D(L). \]

This is sometimes called \textit{Essentially Self-Adjoint}.

(b) If \( \lambda \) is a eigenvalue of a symmetric operator \( L \) then it is real.

(c) If \( f_1, f_2 \) are eigenvectors for a symmetric operator \( L \) for eigenvalues \( \lambda_1 \neq \lambda_2 \), then \( f_1 \perp f_2 \).

(d) If, \( L \) is a symmetric operator with an orthonormal basis of eigenvectors \( \{f_j\} \) and eigenvalues \( \{\mu_j\} \). Then for every \( f \in D(L) \) one has

\[ Lf = L \left( \sum_n \langle f, f_n \rangle f_n \right) = \sum_n \langle Lf, f_n \rangle f_n = \sum_n \mu_n \langle f, f_n \rangle f_n = \sum_n \langle f, f_n \rangle Lf_n. \]

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(e) Examples include many differential operators $L$ which are also symmetric and even self-adjoint. A densely defined operator $L$ with domain $D(L)$ is said to be Self-Adjoint if $D(L) = D(L^*)$ and
\[
(Lf, g) = (f, L^*g) \quad \text{for all } f, g \in D(L).
\]
Here $L$ is typically unbounded so that $L^*$ is also unbounded.

(f) As an example suppose that $L$ is a weighted sum of projections in the separable Hilbert space $H$ given by
\[
Lf = \sum_n \lambda_n P_n f.
\]
then the domain of $L$ is given by
\[
D(L) = \left\{ f \in H : \lim_{N \to \infty} \sum_{k=1}^N \lambda_k P_k f \text{ exists} \right\}.
\]
Then we have
\[
L^* f = \sum_n \overline{\lambda_n} P_n,
\]
and
\[
D(L^*) = D(L).
\]
Recall, $R(P_n)$ are mutually orthogonal and
\[
H = R(P_1) \oplus R(P_1) \oplus \cdots.
\]
Without loss of generality assume that each $R(P_n)$ are all one dimensional. That is, we assume that the eigenvectors of $L$, $\{v_n\}$ form an orthonormal basis for $H$. Let the corresponding eigenvalues be $\{\lambda_n\}$.

(g) In this case we can write
\[
Lf = \sum_n \lambda_n \langle f, v_n \rangle v_n.
\]
then the domain of $L$ is given by
\[
D(L) = \left\{ f \in H : \lim_{N \to \infty} \sum_{k=1}^N \lambda_k \langle f, v_n \rangle v_n \text{ exists} \right\}.
\]
We also have
\[
L^* f = \sum_n \overline{\lambda_n} \langle f, v_n \rangle v_n.
\]
and

\[ D(L^*) = D(L). \]

In this case we can give an alternate characterization of the domain as

\[ D(L) = \left\{ f \in H : \sum_{k=1}^{\infty} |\lambda_k|^2 |\langle f, v_n \rangle|^2 < \infty \right\}. \]

**Unbdd 23** Let \( H \) be a Hilbert space and \( A \in \mathcal{L}(H) \) (linear operator) be self-adjoint (note I am not specifying that \( A \) be bounded). The the following hold:

(a) \( A \) has no residual spectrum.

(b) \( \sigma(A) \subset \mathbb{R} \).

(c) Eigenvectors corresponding to different eigenvalues are orthogonal.

(d) Notice that \( \{ \lambda : \text{Im} (\lambda) \neq 0 \} \subset \rho(A) \). If there is a single \( \lambda \in \rho(A) \) (and hence all) such that \( (\lambda - A)^{-1} \) is compact then we can apply the spectral theorem discussed earlier in weighted sum of projections (see **Spectral Theorem 1** and **Spectral Theorem 2**).

**Grn’s Fn 1** We are interested in in solving problems like

\[ Ly := (py')' - qy = f \]  
\[ B_1 y = \beta_1 y(a) + \gamma_1 y'(a) \]  
\[ B_2 y = \beta_2 y(b) + \gamma_2 y'(b). \]

To this end we define the operator

\[ Ly = (py')' - qy \]

under the assumption that \( \lambda = 0 \) is not an eigenvalue of \( L \) and where \( p, p', \) and \( q \) are continuous on \([a, b]\), \( p(x) > 0 \) on \([a, b]\) and \( |\gamma_j| + |\beta_j| \neq 0 \) for \( j = 1, 2 \), in the Hilbert space \( H = L^2(a, b) \) with inner product

\[ \langle f, g \rangle = \int_a^b f(x)g(x) \, dx \]

and the induced norm

\[ \| f \|^2 = \int_a^b |f(x)|^2 \, dx. \]
For this equation the well known Abel formula for the Wronskian is

$$W(x) = W(a) \frac{p(a)}{p(x)}, \quad (4)$$

Under our assumption that $\lambda = 0$ is not an eigenvalue of $L$, it is always possible to find a basis of solutions $u_j$ of $Lu = 0$ satisfying $B_j u_j = 0$ and $B_i u_j = 0$ for $i = 1, 2$ and $i \neq j$.

A Green’s function for (1)-(3) is a function $g(x, \xi)$ for $(x, \xi) \in [a, b] \times [a, b]$ such that

(a) The following hold

i. $g(\cdot, \cdot)$ is continuous on $[a, b] \times [a, b]$,

ii. $\frac{\partial g}{\partial x}(\cdot, \xi)$ is continuous on $[a, \xi) \times (\xi, b]$, and,

iii. $\frac{\partial g(x, \xi)}{\partial x} \bigg|_{x=\xi^-} = \frac{\partial g(\xi^+, \xi)}{\partial x} - \frac{\partial g(\xi^-, \xi)}{\partial x} = \frac{1}{p(\xi)}$

(b) for all $\xi \in [a, b]$, $g(x, \xi)$ solves $L(g) = 0$, $x \neq \xi$.

(c) for all $\xi \in (a, b)$, $B_i(g) = 0$.

We construct the Green’s function and then we will show that it does indeed lead to a formula for the inverse of $L$.

$$g(x, \xi) = \begin{cases} u_1(x)u_2(\xi), & a \leq x \leq \xi \\ \frac{1}{p(a)W(a)} & \\ u_1(\xi)u_2(x), & \xi < x < b \end{cases} \quad (5)$$

where $Lu_j = 0$ satisfy $B_j u_j = 0$ and $B_k u_j \neq 0$ for $j \neq k$.

The operator $K$ on $H$ defined by

$$K \varphi = \int_a^b g(x, \xi) \varphi(\xi) \, d\xi$$

is a compact operator. Furthermore, it is self-adjoint since

$$g(x, \xi) = g(\xi, x).$$

This is easily verified using the Hilbert-Schmidt theory of compact operators: If $k(x, t)$ satisfies

$$\int_a^b \int_a^b |k(x, \xi)|^2 \, d\xi \, dx < \infty$$

then the operator $K$

$$K \varphi = \int_a^b k(x, \xi) \varphi(\xi) \, d\xi$$

is a called a Hilbert-Schmidt operator and it is a compact operator in $L^2(a, b)$. 

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We now turn to the main application of Green’s function in this section. Namely, we consider the nonhomogeneous BVP.

\[ L_\lambda(y) = (py')' - q(x)y + \lambda y = f(x), \quad a < x < b \]

\[ B_1(y) = 0, \quad B_2(y) = 0 \]

where

\[ B_1y = \beta_1y(a) + \gamma_1y'(a), \quad B_2y = \beta_2y(b) + \gamma_2y'(b), \]

and \( k \in C^1(a,b), p(x) > 0, x \in [a,b]. \)

Namely, given any two functions \( u \) and \( v \), a straightforward calculation gives the so-called Lagrange Identity:

\[ vL_\lambda(u) - uL_\lambda(v) = \frac{d}{dx}P(u,v) \]

where

\[ P(u,v) = p(u'v - uv') \]

and we note that integration gives the so-called Green’s formula

\[ \int_a^b [vL_\lambda(u) - uL_\lambda(v)] = P(u,v)|_{x=a}^{x=b} \]

Let \( g(x,\xi) \) denote the green’s function for the homogeneous problem (1)-(3). From Lagrange’s identity, for \( x \neq \xi \)

\[ g(x,\xi)L_\lambda(y) - yL_\lambda(g(x,\xi)) = \frac{d}{dx}[p(gy' - yg')] \]

which implies

\[ \int_a^{\xi^-} gL_\lambda(y)dx = p(gy' - yg')|_{a}^{\xi^-}_a \]

and

\[ \int_{\xi^+}^b gL_\lambda(y)dx = p(gy' - yg')|_{\xi^+}^{b} \]

Hence

\[ \int_a^b gL_\lambda(y)dx = p(gy' - yg')|_{\xi^-}^{\xi^+} - p(gy' - yg')|_{a}^{b} \]

Note that our boundary conditions \( B_1, B_2 \) have the property that if \( u, v \) satisfy \( B_1(u) = 0 = B_2(v) \), then

\[ [p(gy' - yg')]|_{a}^{b} = 0 \]
Thus we have
\[
\int_a^b g L_\lambda(y) \, dx = -[p(gy' - g'y)]_{\xi^+}^{\xi^-}
= p \left[ \frac{\partial g}{\partial x}(\xi^+, \xi) - \frac{\partial g}{\partial x}(\xi^-, \xi) \right] y(\xi)
= y(\xi).
\]

Therefore if $L^{-1}f = y$ then $y$ satisfies $L(y) = f$, and we have
\[
L^{-1}_{\lambda} f(x) = y(x) = \int_a^b g(x, \xi) f(\xi) \, d\xi.
\]

**Grn’s Fn 11** Since $H$ is a separable Hilbert space we know that there can be at most a countable number of orthogonal eigenvectors. We want to argue that there must exist a $\lambda_0 \in \rho(L)$ so that all the above works. In particular, we can see that $L$ is symmetric in $H$ so by **Unbdd 20** part (c) the eigenvectors of $L$ are orthogonal. Thus there can be at most countable many such eigenfunctions. Thus we can easily find a real $\lambda_0$ so that $L^{-1}_{\lambda_0}$ is a compact self-adjoint operator in $H$.

Thus we have the following result:

For a Regular Sturm-Liouville operator $L$ there exists a sequence of real numbers $\{\mu_j\}$ (which tend to minus infinity) and a corresponding orthonormal basis of eigenfunctions $\{v_j\} \subset C^2[a, b]$ satisfying $Lv_j = \mu_j v_j$. Furthermore, we have the following generalized Fourier series
\[
u = \sum_{j=1}^{\infty} \langle u, v_j \rangle v_j \quad \forall \ u \in H.
\]

\[
u \in D(L) \iff \sum_{j=1}^{\infty} |\langle u, v_j \rangle|^2 < \infty,
\]

and for $u \in D(L)$ we have
\[
Lu = \sum_{j=1}^{\infty} \mu_j \langle u, v_j \rangle v_j.
\]

**Semi-Grp 1** A one-parameter family $T(t)$ for $0 \leq t < \infty$ of bounded linear operators on a Banach space $X$ is a $C_0$ (or strongly continuous) Semigroup on $X$ if

(a) $T(0) = I$ (the identity on $X$).

(b) $T(t + s) = T(t)T(s)$ (semigroup property)
(c) \(\lim_{t \to 0} T(t)v = v\) for all \(v \in X\) (This the Strong Continuity at \(t = 0\), i.e., continuous at \(t = 0\) in the strong operator topology).

**Semi-Grp 2** If \(T(t)\) is a \(C_0\) semigroup, then there exists \(\omega \geq 0\) and \(M \geq 1\) such that

\[
\|T(t)\| \leq Me^{\omega t} \quad \forall \ t \geq 0.
\]  

(6)

**Semi-Grp 3** If \(T(t)\) is a \(C_0\) semigroup then for every \(v \in X\), \(T(t)v\) is continuous on \(\mathbb{R}_+ = [0, \infty)\) into \(X\).

**Semi-Grp 4** The Infinitesimal Generator of \(T(t)\) is the linear operator \(A\) defined as follows

\[
D(A) = \left\{ v \in X : \lim_{t \to 0} \frac{T(t)v - v}{t} \text{ exists} \right\}
\]  

and for \(v \in D(A)\) we define

\[
Av = \lim_{t \to 0} \frac{T(t)v - v}{t}.
\]  

(8)

**Semi-Grp 5** Let \(T(t)\) be a \(C_0\) semigroup and \(A\) its infinitesimal generator. Then we have

(a) for \(v \in X\)

\[
\lim_{h \to 0} \frac{1}{h} \int_{t}^{t+h} T(s)v \, ds = T(t)v.
\]  

(9)

(b) For every \(v \in X\), \(\int_{0}^{t} T(s)v \, ds \in D(A)\) and

\[
A \left( \int_{0}^{t} T(s)v \, ds \right) = T(t)v - v.
\]  

(10)

(c) for every \(v \in D(A)\), \(T(t)v \in D(A)\) and

\[
\frac{d}{dt} T(t)v = AT(t)v = T(t)Av.
\]  

(11)

(d) for every \(v \in D(A)\),

\[
T(t)v - T(s)v = \int_{s}^{t} T(\tau)Av \, d\tau = \int_{s}^{t} AT(\tau)v \, d\tau.
\]  

(12)

**Semi-Grp 6** If \(A\) is the infinitesimal generator of a \(C_0\) semigroup \(T(t)\), then \(D(A)\) is dense in \(X\) and \(A\) is a closed linear operator.

**Semi-Grp 7** Let \(T(t)\) and \(S(t)\) be two \(C_0\) semigroups with infinitesimal generator \(A\) and \(B\) respectively.

If \(A = B\) then \(T(t) = S(t)\) for all \(t\).
(a) The growth bound of a $C_0$ semigroup $T(t)$ is defined to be
\[ \omega_0 = \inf_{t>0} \left( \frac{\ln(\|T(t)\|)}{t} \right). \] (13)

(b) A function $w : \mathbb{R}_+ \to \mathbb{R}$ is called sub-additive if
\[ w(t_1 + t_2) \leq w(t_1) + w(t_2). \]

Semi-Grp 9 If $w$ is sub-additive and bounded above on any finite subinterval, then
\[ w_0 = \inf_{t>0} \frac{w(t)}{t} \text{ is finite, or } -\infty, \]
and
\[ w_0 = \lim_{t \to \infty} \frac{w(t)}{t}. \]

Semi-Grp 10 If $T(t)$ is a $C_0$ semigroup in $X$ the the growth bound $\omega_0$ is given by
\[ \omega_0 = \lim_{t \to \infty} \left( \frac{\ln(\|T(t)\|)}{t} \right), \] (14)
and for every $\omega > \omega_0$ there exists a $M_\omega$ such that, for all $t \geq 0$
\[ \|T(t)\| \leq M_\omega e^{\omega t}. \] (15)

Semi-Grp 11 Let us use the notation
\[ R_\lambda(A) = (\lambda - A)^{-1} \] (16)
to denote the resolvent operator for $\lambda \in \rho(A)$.

Semi-Grp 12 If $T(t)$ is a $C_0$ semigroup in $X$ with growth bound $\omega_0$. Let $\text{Re} (\lambda) > \omega > \omega_0$. Then $\lambda \in \rho(A)$ and for every $v \in X$
\[ R_\lambda(A)v = \int_0^{\infty} e^{-\lambda t}T(t)v \, dt \] (17)
and
\[ \|R_\lambda(A)\| \leq \frac{M_\omega}{(\sigma - \omega)} \text{ where } \sigma = \text{Re} (\lambda). \] (18)

Semi-Grp 13 If $A$ is a closed densely defined operator such that for every $\alpha > \omega$
\[ \|R_\alpha(A)\| \leq \frac{M_\omega}{\alpha - \omega}, \quad \alpha > \omega. \] (19)
then
\[ \lim_{\alpha \to \infty} \alpha R_\alpha(A)v = \lim_{\alpha \to \infty} \alpha (\alpha - A)^{-1}v = v \quad \forall \ v \in X. \] (20)
Semi-Grp 14 (Hille-Yoshida Theorem) A linear operator $A$ acting in a Banach space $X$ is the infinitesimal generator of a $C_0$ semigroup $T(t)$ such that $\|T(t)\| \leq Me^{\omega t}$ if, and only if

1. $A$ is closed and $\overline{D(A)} = X$
2. $\|R_\lambda(A)^n\| \leq \frac{M}{(\lambda - \omega)^n}, \quad \lambda > \omega, \quad n = 1, 2, 3, \ldots$.

Semi-Grp 15 As a final example of operators $T$ which are given by a weighted sum of projections suppose that $T$ is a weighted sum of projections in a separable Hilbert space $H$ and $R(P_n)$ are all one dimensional so that the eigenvectors $\{v_j\}$ form an orthonormal basis. Thus we can write

$$Lf = \sum_n \lambda_n \langle f, v_j \rangle v_j.$$ 

For example, $L$ could be a regular Sturm-Liouville operator.

Then $L$ is the infinitesimal generator of a $C_0$ semigroup which is given by

$$T(t)f = \sum_{k=1}^{\infty} e^{\mu_k t} \langle f, v_k \rangle v_k.$$ 

More generally, if $T$ which are given by a weighted sum of projections has representation, in terms of a Resolution of the Identity $\{P_j\}$ with eigenvalues $\{\mu_j\}$, then $T$ generates a $C_0$ semigroup given by

$$T(t)f = \sum_{j=1}^{\infty} e^{\mu_j t} P_j f.$$ 

Sobolev 1 The course ended with two lectures from Notes I wrote on Sobolev Spaces on the circle (compact manifold without boundary). We covered many basic results such as:

(a) Review of basic Fourier series.

(b) Let $0 \leq p \leq \infty$. Then by $H^p[0, 2\pi]$ we denote

$$H^p[0, 2\pi] = \left\{ \varphi \in L^2[0, 2\pi] : \sum_{m=-\infty}^{\infty} (1 + m^2)^p |a_m|^2 < \infty \right\}$$

where $\{a_m\}$ are the Fourier coefficients of $\varphi$. The space $H^p[0, 2\pi]$ is called a Sobolev Space. Note that

$$H^0[0, 2\pi] = L^2[0, 2\pi].$$

(c) Sobolev Embedding If $\varphi \in H^p[0, 2\pi]$ and $p \geq k + 1/2$ then $\varphi \in C^k$ and

$$\max_{x \in [0, 2\pi]} |D^\ell \varphi| \leq \|u\|^p \quad \text{for} \quad \ell \leq k.$$ 

Note, for example, that for every $p \geq 1$ and $k = 0$ we have $\varphi \in C[0, 2\pi]$. 

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(d) For $0 < p < \infty$ we denote by $H^{-p}[0, 2\pi]$ the dual space of $H^p[0, 2\pi]$, i.e., the space of bounded linear functionals on $H^p[0, 2\pi]$. 

(e) i. $|\langle \phi, \psi \rangle_p| \leq \| \phi \|_p \| \psi \|_p$

ii. $|\langle \phi, \psi \rangle_p| \leq \| \phi \|_{p+q} \| \psi \|_{p-q}$ or with $\varphi = \psi$ the $\| \varphi \|_p^2 \leq \| \varphi \|_{p+q} \| \varphi \|_{p-q}$

iii. $\| \varphi \|_p \leq \| \varphi \|_q$ for all $p < q$.

iv. For every $\epsilon > 0$ and $p_1 > p > p_2$ we have

$$\| \varphi \|_p \leq \epsilon \| \varphi \|_{p_1} + \epsilon^{-(p-p_2)/(p_1-p)} \| \varphi \|_{p_2}. $$

v. If $D = d/dt$, $\varphi = \sum_{m=-\infty}^{\infty} a_m e^{imt}$, then

$$\| D^p \varphi \|_q \leq \| \varphi \|_{p+q}. $$

More generally, there exists constants $M_1$ and $M_2$ such that

$$M_1 \| \varphi \|_p \leq \| \varphi \| \leq M_2 \| \varphi \|_p \quad \text{for all } \varphi \in H^p[0, 2\pi]. \quad (22)$$

vi. $\| \varphi \|_p \leq C \sum_{0 \leq k \leq p} \| D^k \varphi \|_0$ for some constant $C$ independent of $\varphi$.

vii. If $\eta \in C^\infty[0, 2\pi]$ and $\varphi \in H^p[0, 2\pi]$, then $\eta \varphi \in H^p[0, 2\pi]$ and

$$\| \eta \varphi \|_p \leq C \| \varphi \|_p.$$  

References


