

1. Determine the radius of convergence of the power series  $\sum_{m=1}^{\infty} \frac{4^n}{n^2} x^{2n}$

**ANSWER:**

$$\sum_{m=1}^{\infty} \frac{4^n}{n^2} x^{2n} = \sum_{m=1}^{\infty} \frac{(4x^2)^n}{n^2} \text{ so set } t = 4x^2 \text{ to obtain } \sum_{m=1}^{\infty} \frac{t^n}{n^2}. \text{ Thus, for } t,$$

we have

$$R = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n \rightarrow \infty} \frac{(n+1)^2}{n^2} = 1$$

so the series converges for  $|4x^2| = |t| < 1$  which means for  $|x| < 1/2$ .

2. Consider the equation:  $(x-3)y' + 2y = 0$

- (a) Find the general solution in the form  $y = \sum_{n=0}^{\infty} a_n x^n$

**ANSWER:**

$$y = c_0 \sum_{n=0}^{\infty} \frac{(n+1)}{3^n} x^n.$$

- (b) Find the radius of convergence of the series.

**ANSWER:**

$$y = \sum_{n=0}^{\infty} (n+1) \left(\frac{x}{3}\right)^n \stackrel{t=x/3}{=} \sum_{n=0}^{\infty} (n+1)t^n \text{ so}$$

$$R = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n \rightarrow \infty} \frac{n+1}{n+2} = 1$$

and the series converges for  $|t| < 1$  or  $|x| < 3$ .

- (c) Sum the series to obtain a “closed form solution.” (hint: use a geometric series formula).

**ANSWER:**

To sum the series we note that

$$\sum_{n=0}^{\infty} (n+1)t^n = \frac{d}{dt} \left( \sum_{n=0}^{\infty} t^{n+1} \right) = \frac{d}{dt} \left( t \sum_{n=0}^{\infty} t^n \right) = \frac{d}{dt} \left( \frac{t}{1-t} \right) = \frac{1}{(1-t)^2}.$$

$$\text{So we get } y = \frac{9c_0}{(3-x)^2}.$$

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3. Find a recursion formula for a solution of  $(x^2 - 1)y'' + 2xy' + 2xy = 0$  in the form  $y = \sum_{m=0}^{\infty} a_m x^m$ . Then find the first three nonzero terms in each of two linearly independent solutions.

**ANSWER:**

$$c_2 = c_3 = 0, \quad c_{n+4} = -\frac{(n+1)c_{n+1} - c_n}{(n+3)(n+4)} \text{ for } n \geq 0.$$

$$y_1 = 1 - \frac{1}{12}x^4 + \frac{1}{126}x^7 + \dots, \quad y_2 = x - \frac{1}{12}x^4 - \frac{1}{20}x^5 + \dots.$$

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4. Use the Taylor method to obtain the first three nonzero terms in a series solution for the equation  $y' = \sin(y) + e^x$ ,  $y(0) = 0$ .

**ANSWER:**

We seek  $y = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{y^{(n)}(0)}{n!} x^n$ . Then we have  $y(0) = 0$  implies that  $a_0 = 0$ . Then  $y' = \sin(y) + e^x$  implies  $y'(0) = \sin(0) + e^0 = 1$  so  $a_1 = 1$ . Next, we differentiate the equation to obtain  $y'' = y' \cos(y) + e^x$  so that  $y''(0) = 1 + 1 = 2$  and  $a_2 = \frac{2}{2} = 1$ . Finally, we differentiate again to get  $y''' = y'' \cos(y) - (y')^2 \sin(y) + e^x$  so that  $y'''(0) = 2 + 1 = 3$  and  $a_3 = \frac{3}{3!} = \frac{1}{2}$ . Thus we have

$$y = x + x^2 + \left(\frac{1}{2}\right)x^3 + \dots.$$

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5. Investigate the nature of the possible singular point at  $x = 0$  for the equations. Also find the indicial equation and describe the possible form of the solutions.

(a)  $x^2 y'' + \cos(x)y' + xy = 0$

**ANSWER:**

$x = 0$  is an irregular singular point. Nothing we can say.

(b)  $x^2(x+2)y'' - xy' + (1+x)y = 0$

**ANSWER:**

The indicial equation reduces to  $2r^2 - 3r + 1 = 0$  so we get  $r_1 = 1$  and  $r_2 = 1/2$ . Therefore  $y_1 = x \sum_{n=0}^{\infty} a_n x^n$  (with  $a_0 \neq 0$ ), and

$$y_2 = x^{1/2} \sum_{n=0}^{\infty} A_n x^n \text{ (with } A_0 \neq 0)$$

(c)  $x(1+x)y'' + 2y' + 3xy = 0$

**ANSWER:**

The indicial equation reduces to  $r^2 + r = 0$  so we get  $r_1 = 0$  and  $r_2 = -1$ . Since these roots differ by an integer so we have one solution (note  $r_2 = 0$  is the larger root)  $y_1(x) = \sum_{n=0}^{\infty} a_n x^n$  (with  $a_0 \neq 0$ ), and

$$y_2(x) = ky_1(x) \ln(x) + x^{-1} \sum_{n=0}^{\infty} A_n x^n \text{ (with } A_0 \neq 0 \text{)} .$$

6. Find the Frobenius series solutions of

(a)  $4xy'' + 2y' + y = 0$

**ANSWER:**

$$y_1 = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n)!} = \cos(\sqrt{x}),$$

$$y_2 = x^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{(2n+1)/2}}{(2n+1)!} = \sin(\sqrt{x}).$$

(b)  $4xy'' + 8y' + xy = 0$

**ANSWER:**

The indicial equation is  $r^2 + r = 0$  so  $r_1 = 0$  and  $r_2 = -1$ . Thus we are in case 3. After lots of work we find:

$$y_1 = \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n}}{2(2n+1)!} = \frac{\sin(x/2)}{x}$$

$$y_2 = x^{-1} \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n}}{(2n)!} = \frac{\cos(x/2)}{x}.$$