TOPOLOGY DOCTORAL PRELIMINARY EXAMINATION
MAY 2003

WORK ALL PROBLEMS. ASSUME THAT ALL SPACES UNDER CONSIDERATION ARE HAUSDORFF (T_2). GIVE A PRECISE STATEMENT OF ANY MAJOR THEOREM REFERENCED IN ANY ARGUMENT.

1.) Let $f : X \twoheadrightarrow Y$ be a closed, continuous surjection with X locally connected. Show that Y is also locally connected.

2.) Let X be a topological space and let (Y, d) be a metric space. Show that the set $\mathcal{C}(X, Y)$ of all continuous functions from X to Y is closed in the space Y^X of all functions from X to Y under the uniform metric. Use this result to show that if Y is complete under the metric d, then $\mathcal{C}(X, Y)$ is complete under the corresponding uniform metric.

3.) Let $f : X \to \prod_{\alpha\in J} X_\alpha$ be given by the equation $f(x) = (f_\alpha(x))_{\alpha\in J}$, where $f_\alpha : X \to X_\alpha$ for each α in an arbitrary index set J. Let $\prod_{\alpha\in J} X_\alpha$ have the standard product topology. Show that f is continuous if and only if each function f_α is continuous.

Give an example to show that there exists a space X and function $f : X \to \prod_{\alpha\in J} \mathbb{R}_j$ given by $f(x) = (f_\alpha(x))_{\alpha\in J}$, where each $f_\alpha : X \to \mathbb{R}_\alpha$ is continuous and $\prod_{\alpha\in J} \mathbb{R}_\alpha$ has either the uniform topology or the box topology, but f is not continuous.

4.) A space X is *locally metrizable* if each point of x has a neighborhood which is metrizable in the subspace topology. Show that a regular Lindelöf space is metrizable if it is locally metrizable.

5.) Consider the product space $X \times Y$, where Y is compact. If U is an open set in $X \times Y$ containing the set $\{x_0\} \times Y$, show that there exists a "tube" $V \times Y$ such that $\{x_0\} \times Y \subset V \times Y \subset U$, where V is an open set in X containing x_0.

6.) Let $p : E \to B$ be a covering map, with $p(e_0) = b_0$. Let f and g be paths in B from b_0 to b_1, with $\tilde{f}$ and $\tilde{g}$ their respective liftings to paths in E beginning at e_0. Show that if f and g are path homotopic in B then $\tilde{f}$ and $\tilde{g}$ end at the same point of E and are path homotopic in E.

7.) Prove the equivalence of the following two statements. Prove one of the statements.
a.) There is no retract of a disk to its boundary.
b.) Every continuous function from a disk to itself has a fixed point.

8.) Assume that each of U, V and $U \cap V$ is an arcwise-connected open subset of the space X, where $X = U \cup V$ and $x_0 \in U \cap V$. Assume that V is simply connected. Show that $\psi_1 : \pi_1(U, x_0) \to \pi_1(X, x_0)$ is an epimorphism and its kernel is the smallest normal subgroup of $\pi_1(U, x_0)$ containing the image $\phi_1(\pi_1(U \cap V, x_0))$, where ψ_1 is the homomorphism induced by the inclusion of U into X and ϕ_1 is the homomorphism induced by the inclusion of $U \cap V$ into U. (This is a special case of the Seifert-van Kampen Theorem. Do not simply quote this theorem.)

Use this result to give a representation of the fundamental group of the Klein bottle.