2 Discrete and Continuous Dynamical Systems

2.1 Linear Discrete and Continuous Dynamical Systems

There are two kinds of dynamical systems: discrete time and continuous time. For a discrete time dynamical system, we denote time by \( k \), and the system is specified by the equations

\[
x(0) = x_0, \quad \text{and} \quad x(k + 1) = f(x(k)).
\]

It thus follows that \( x(k) = f_k(x_0) \), where \( f^k \) denotes a \( k \)-fold application of \( f \) to \( x_0 \). For a continuous time dynamical system, we denote time by \( t \), and the following equations specify the system:

\[
x(0) = x_0, \quad \text{and} \quad \dot{x} = f(x).
\]

We have seen that one way to arrive at a discrete dynamical system is applying Euler’s (or some other numerical scheme) to a continuous dynamical system.

We have introduced the ideas of discrete and continuous time dynamical systems and we hope it is clear that the notion of a dynamical system can be useful in modeling many different kinds of phenomena. Once we have created a model, we would like to use it to make predictions. Given a dynamical system either of the discrete form \( x(k+1) = f(x(k)) \) or of the continuous sort \( \dot{x} = f(x) \), and an initial value \( x_0 \), we would very much like to know the value of \( x(k) \) [or, \( x(t) \)] for all values of \( k \) [or \( t \)]. In some rare instances, this is possible. For example, if \( f \) is a linear function. Unfortunately, it is all too common that the dynamical system in which we are interested does not yield an analytic solution. What then? One option is

*Most of the material in this section is taken from the outstanding internet version of the book *Invitation to Dynamical Systems* by Ed Scheinerman*
numerical methods. However, we can also determine the qualitative nature of the solution. We have explored the notion of fixed points and found that for autonomous one dimensional continuous dynamical systems the long time behavior of the system is completely determined by the fixed points. Namely using the fixed points (equilibria) we can draw the phase line which completely determines the asymptotic behavior of solutions. For higher dimensional systems (two and higher) there are much more interesting dynamics for example we can encounter behaviors like periodicity. But there are even other types interesting behavior including blow up (its state vector goes to infinity with time) or chaotic behavior,

2.1.1 Linear Discrete Dynamical Systems

In this section we study dynamical systems in which the function $f$ is linear, i.e.,

$$f(x) = ax + b,$$

where $a$ and $b$ are constants. Thus we consider

$$x(k+1) = ax(k) + b; \quad x(0) = x_0.$$  \hspace{1cm} (1)

We discuss this case first analytically (i.e., by equations) and then geometrically (with graphs).

Suppose first that $b = 0$, i.e., $x(k+1) = ax(k)$. It is very clear that for any $k$ we have simply that $x(k) = a^k x_0$.

1. If $|a| < 1$, then $a^k \to 0$ as $k \to \infty$ and so $x(k) \to 0$.

2. If $|a| > 1$, then $a^k \to \infty$ as $k \to \infty$. Thus unless $x_0 = 0$, we have $|x(k)| \to \infty$.

3. If $a = 1$, then $x(0) = x(1) = x(2) = \cdots$ so $x(k) = x_0$ for all $k$, i.e, $x_0$ is a fixed point.

4. If $a = 1$, then $x(0) = -x(1) = x(2) = \cdots$ so $x(k) = (-1)^k x_0$ for all $k$, i.e.e the solution oscillates.
Next we consider the more general case in (1). We begin by working out the first few values:

\[ x(0) = x_0 \]
\[ x(1) = ax(0) + b = ax_0 + b \]
\[ x(2) = ax(1) + b = a(ax_0 + b) + b = a^2x_0 + b(a + 1) \]
\[ x(3) = ax(2) + b = a(a^2x_0 + (ab + b)) + b = a^3x_0 + b(a^2 + a + 1) \]
\[ x(4) = ax(3) + b = a(a^3x_0 + b(a^2 + a + 1)) + b = a^4x_0 + b(a^3 + a^2 + a + 1) \]

From this the pattern is clear

\[ x(k) = a^kx_0 + b \sum_{j=0}^{k-1} a^j \]

and applying the formula for a geometric series we have

\[ x(k) = \begin{cases} 
  a^kx_0 + b \left( \frac{a^k-1}{a-1} \right), & a \neq 1 \\
  a^kx_0 + kb, & a = 1 
\end{cases} \tag{2} \]

1. Now for \(|a| < 1\) we have \(a^k \to 0\) as \(k \to \infty\). Thus

\[ x(k) \xrightarrow{k \to \infty} \frac{b}{1-a} \equiv \bar{x}. \]

Note \(f(\bar{x}) = \bar{x}\) as we can see from

\[ f(\bar{x}) = a \left( \frac{b}{1-a} \right) + b = \frac{ab + (1-a)b}{1-a} = \frac{b}{1-a} = \bar{x}. \]

Thus \(\bar{x}\) is a stable fixed point or equilibrium (i.e., an attractor) since the solutions are attracted to it.
2. If $|a| > 1$ then $|a|^k \to \infty$ as $k \to \infty$. In this case we can write (2) as

$$x(k) = a^k \left( x_0 - \frac{b}{a-1} \right) + \left( \frac{b}{a-1} \right).$$

From this we see that if then

$$x_0 \neq \frac{b}{a-1} \Rightarrow |x(k)| \xrightarrow{k \to \infty} \infty, \quad \text{and} \quad x_0 = \frac{b}{a-1} \Rightarrow x(k) = \overline{x} \quad \text{for all} \quad k.$$

3. Next we consider $a = 1$. in which case from (2) we see that $a^k x_0 + k = x_0 + kb$ so if $b \neq 0$ then $|x(k)| \to \infty$ and, otherwise, (if $b = 0$ ) then $x(k) = x_0$ for all $k$, i.e., the solution stays at the initial condition for all $k$.

4. Finally, if $a = -1$ then

$$x(0) = x_0$$
$$x(1) = -x_0 + b$$
$$x(2) = x_0 + b(-1 + 1) = x_0$$
$$x(3) = -x_0 + b$$
$$x(4) = x_0$$

so $x(k)$ oscillates between $x_0$ and $b - x_0$. There is one special case in which $x_0 = b - x_0$ which implies

$$x_0 = \frac{b}{2} = \frac{b}{1-(-1)} = \frac{b}{1-a} = \overline{x}.$$

This is the same fixed point we saw earlier.
Iterating $f(x) = ax + b$ with $0 < a < 1$. 

Iterating $f(x) = ax + b$ with $-1 < a < 0$. 

Iterating $f(x) = ax + b$ with $a > 1$. 

Iterating $f(x) = ax + b$ with $a < -1$.

Iterating $f(x) = ax + b$ with $a = 1$.

Iterating $f(x) = ax + b$ with $a = -1$. 
2.1.2 Linear Continuous Dynamical Systems

In the case of a continuous time system we have

\[ x(0) = x_0, \quad \text{and} \]
\[ \dot{x} = ax + b. \]

with solution given by

\[ x(t) = e^{at} \left( x_0 + \frac{b}{a} \right) - \frac{b}{a}. \]

1. If \( a < 0 \), then \( e^{at} \to 0 \) as \( t \to \infty \). Thus \( x(t) \to -b/a \) regardless of the value of \( x_0 \). We call \( \bar{x} = -b/a \) a stable fixed point of the system. It is called fixed because if the system is in state \( \bar{x} \), then it will be there for all time. It is stable because the system moves toward that value as \( t \) goes to infinity.

2. If \( a > 0 \) then \( e^{at} \to \infty \) as \( t \to \infty \). Therefore, unless \( x_0 = -b/a \) the solution goes to infinity. If \( x_0 = -b/a \) then the solution stays at this value for every - it is a fixed point and we say it is an unstable fixed point.

3. Finally if \( a = 0 \) then \( x(t) = bt + x_0 \) so if \( b = 0 \) the system is stuck at \( x_0 \) but if \( b \neq 0 \) then \( x(t) \) goes to infinity for every \( x_0 \).

Iterating \( \dot{x} = ax + b \) with \( a < 0 \).
Iterating $\dot{x} = ax + b$ with $a > 0$.

Iterating $\dot{x} = ax + b$ with $a = 0$.

Exercises

1. Find an exact formula for $x(k)$, if $x(k + 1) = ax(k) + b$, $x(0) = x_0 = c$, and $a$, $b$, and $c$ have the following values:
   
   (i) $a = 1$, $b = 1$, $c = 1$
   
   (ii) $a = 1$, $b = 0$, $c = 2$
   
   (iii) $a = 3/2$, $b = -1$, $c = 0$
   
   (iv) $a = -1$, $b = 1$, $c = 4$
   
   (v) $a = -1/2$, $b = 1$, $c = 3/2$

2. For each of the discrete time systems in the previous problem, determine whether or not $|x(k)| \to \infty$. Determine if the system has a fixed point and whether or not the system is approaching that fixed point.

3. Find an exact formula for $x(t)$, if $\dot{x} = ax + b$, $x(0) = x_0 = c$, and $a$, $b$, and $c$ have the following values:

   (i) $a = 1$, $b = 0$, $c = 1$

   (ii) $a = 0$, $b = 1$, $c = 0$

   (iii) $a = 0$, $b = 0$, $c = 1$
(iv) \( a = -1, \ b = 1, \ c = 2 \)

(v) \( a = 2, \ b = 3, \ c = 0 \)

4. For each of the continuous time systems in the previous problem, determine whether or not \( |x(t)| \to \infty \). Determine if the system has a fixed point and whether or not the system is approaching that fixed point.

2.2 Nonlinear Discrete and Continuous Dynamical Systems

In this section we consider general discrete and continuous systems

\[
x(0) = x_0, \quad \text{and} \quad x(k + 1) = f(x(k)).
\]

It thus follows that \( x(k) = f_k(x_0) \), where \( f^k \) denotes a \( k \)-fold application of \( f \) to \( x_0 \). For a continuous time dynamical system, we denote time by \( t \), and the following equations specify the system:

\[
x(0) = x_0, \quad \text{and} \quad \dot{x} = f(x).
\]

In the last section we closely examined the case when \( f \) is linear, and in that case, we saw that we could answer nearly any question we might consider. We can work out exact formulas for the behavior of \( x(t) \) (or \( x(k) \)) and deduce from them the long-term behavior of the system. There are two main behaviors: (1) the system gravitates toward a fixed point, or (2) the system blows up. There are some marginal behaviors as well.

Now we begin our study of more general systems in which \( f \) can be virtually any function. However, we do make the following assumption:

\[
\text{We assume } f \text{ is differentiable with continuous derivative.}
\]

Will this broad generality make our work more complicated? Yes and no:

Yes: Nonlinear functions can present insurmountable problems. Typically, it is impossible
to find exact formulas for $x$. Further, the range of behaviors available to nonlinear systems is much greater than that for linear systems (but that’s why nonlinear systems are more interesting).

No: Because it can be terribly difficult to find exact solutions to nonlinear systems, we have a valid excuse for not even trying! Instead, we settle for a more modest goal: determine the long-term behavior of the system. This is often feasible even when finding an exact solution is not.

### 2.2.1 Fixed Points and Stability

As we have already learned in earlier chapters for continuous systems, we will focus on the notion of a fixed point (sometimes called an equilibrium point) of a dynamical system. Indeed we have already spent some time talking about fixed points for continuous systems and how to determine if they are stable or unstable. Often, understanding the fixed points of a dynamical system can tell us much about the global behavior of the system. We need to repeat this analysis for discrete systems.

Notice that the notion of a fixed point appears, at first glance, as slightly different in the discrete and continuous cases. Actually it is not different, a fixed point of a dynamical system is a state vector $\bar{x}$ with the property that if the system is ever in the state $\bar{x}$, it will remain in that state for all time.

For a continuous time system $\dot{x} = f(x)$ a vector $\bar{x}$ with the property that the system remains in the state $\bar{x}$ for all time means that it does not depend on time, i.e., $x'(t) = 0$ so that a fixed point corresponds to a value for which $f(x) = 0$. For a discrete system a value $\bar{x}$ is a fixed point if $\bar{x} = f(\bar{x})$.

As we have already seen, not all fixed points are the same: some are stable, some are unstable and some are semi-stable. For nonlinear discrete systems we have not yet talked about these concepts so we address this problem now. We begin by illustrating these concepts with an example. Let $f(x) = x^2$ and consider the discrete time dynamical system

$$x(k+1) = f(x(k)) = [x(k)]^2.$$
The system has two fixed points: 0 and 1 (these are the solutions to \( f(x) = x \) or \( x^2 = x \)). You will notice that if \( x_0 = 0 \) or 1 then \( x(k) \) will remain that value for ever. If we take a different \( x_0 \) the we arrive at the sequence

\[
x_0 \mapsto x_0^2 \mapsto x_0^4 \mapsto x_0^8 \mapsto \cdots x_0^{2^k}.
\]

Thus if \( |x_0| \) is less than one the \( x(k) \) go to zero so we say \( x = 0 \) is a stable equilibrium. On the other hand if \( |x_0| \) is bigger than one it will increase with outbound. So in any case initial conditions near 1 move away from 1 and we say that \( x = 1 \) is an unstable equilibrium.

**Definition 2.1.** 1. First, a fixed point \( \bar{x} \) is called stable if for all initial values \( x_0 \) near \( \bar{x} \) the solution starting at \( x_0 \) satisfies

\[
x(t) \xrightarrow{t \to \infty} \bar{x} \text{ (continuous case), } x(k) \xrightarrow{k \to \infty} \bar{x} \text{ (discrete case).}
\]

2. A fixed point \( \bar{x} \) is called *marginally stable* (also called neutrally stable) if for all initial values \( x_0 \) near \( \bar{x} \) the solution starting at \( x_0 \) stays near \( x_0 \) but does not converge to \( \bar{x} \).

3. A fixed point \( \bar{x} \) is called *unstable* if it is neither stable nor marginally stable. In other words, there are starting values \( x_0 \) very near \( \bar{x} \) so that the system moves far away from \( \bar{x} \).

The following figure illustrates the three possibilities.

Fixed point 1 unstable, 2 is marginally stable and 3 is stable.
Exercises

1. Find all fixed points of the following discrete time systems $x(k + 1) = f(x(k))$.

   (i) $f(x) = x^2 - 2$
   (ii) $f(x) = \sin(x)$
   (iii) $f(x) = 1/x$
   (iv) $f(x) = \sqrt{x^2}$

2. Do numerical experiments near each of the fixed points you found in the previous problem to determine their stability.

3. Find all fixed points of the following continuous time systems $\dot{x} = f(x)$

   (i) $f(x) = x^2 - x - 1$
   (ii) $f(x) = \sin(x)$
   (iii) $f(x) = e^x - 1$
   (iv) $f(x) = \ln(x^2)$
   (v) $f(x) = x/(1 - x)$

4. Explain why it is impossible for a linear system (either discrete or continuous) to have exactly two fixed points.

5. Use graphical analysis to show that iterating $\cos(x)$ from any starting value $x_0$ always leads to the same answer: the unique fixed point of $x(k + 1) = \cos(x(k))$.

Using a method called linearization one can (almost) classify the stability of fixed points in both the discrete and continuous time systems based on properties of the derivative of $f(x)$ at the fixed point.

Theorem 2.1. 1. Continuous time Let $\pi$ be a fixed point of the continuous time dynamical system $\dot{x} = f(x)$. If $f'(\pi) < 0$, then $\pi$ is a stable fixed point. If $f'(\pi) > 0$, then $\pi$ is a unstable fixed point. If $f'(\pi) = 0$, the test fails, i.e., the fixed point may or may not be stable.
2. Discrete time Let $\bar{x}$ be a fixed point of the discrete time dynamical system $x(k+1) = f(x(k))$. If $|f'(\bar{x})| < 1$, then $\bar{x}$ is a stable fixed point. If $|f'(\bar{x})| > 1$, then $\bar{x}$ is a unstable fixed point. If $|f'(\bar{x})| = 1$, the test fails, i.e., the fixed point may or may not be stable.

Exercises

1. For each of the following discrete time systems $x(k+1) = f(x(k))$, find all fixed points and determine their stability.
   
   (i) $f(x) = -x^3 - 2$
   
   (ii) $f(x) = x^2 - x + 1/4$
   
   (iii) $f(x) = \frac{e^{x/2} - 1}{x}$
   
   (iv) $f(x) = (3/2) \sin(x)$
   
   (v) $f(x) = e^{\cos(x)}$

2. In this problem we consider one-dimensional discrete time dynamical systems $x(k+1) = f(x(k))$ with a fixed point $\bar{x}$ at which $|f'(\bar{x})| = 1$. For each of the following systems, discuss the stability of the fixed point $\bar{x}$.
   
   (i) $f(x) = \sin(x)$, $\bar{x} = 0$
   
   (ii) $f(x) = x^3 + x$, $\bar{x} = 0$
   
   (iii) $f(x) = 1 + \ln(x)$, $\bar{x} = 1$
   
   (iv) $f(x) = x^2 + 1/4$, $\bar{x} = 1/2$
   
   (v) $f(x) = 1/x$, $\bar{x} = 1$

3. In this problem we consider one-dimensional continuous time dynamical systems $\dot{x} = f(x)$ with a fixed point at $\bar{x}$ for which $f'(\bar{x}) = 0$. For each of the following systems, discuss the stability of the fixed point $\bar{x} = 0$.
   
   (i) $f(x) = x^2$
   
   (ii) $f(x) = -x^2$
   
   (iii) $f(x) = x^3 - 3x^2 + 2x$
(iv) \( f(x) = x^3 \)
(v) \( f(x) = -x^3 \)

### 2.3 Periodicity and Chaos

Dynamical systems do not live by fixed points alone. For one dimensional continuous systems pretty much tell the whole story. For two dimensional continuous systems one can also get periodicity and for three dimensional systems one can also have chaotic behavior. But for one dimensional discrete systems all these things are possible.

What is “periodic behavior”? A dynamical system exhibits periodic behavior when it returns to a previously visited state. The system retakes the same steps over and over again, visiting the same states infinitely often. A fixed point is an extreme example of periodic behavior.

What is “chaos”? For now we suffice to say that a system can behave in a nonperiodic and nonexplosive manner which, although completely determined, is utterly unpredictable!

#### 2.3.1 Periodicity for Discrete Systems

Let \( x(k + 1) = f(x(k)) \) be a one-dimensional discrete time dynamical system. We can write \( x(k) = f^k(x) \). A fixed point of this system is a value \( \bar{x} \) for which \( f(\bar{x}) = \bar{x} \). More generally, a periodic point of this system is a value \( \bar{x} \) for which \( f^k(\bar{x}) = \bar{x} \). We call the number \( k \) a period of \( x \). Now if \( x \) is a periodic point with period \( k \) we know that \( x = f^k(x) \), but it then follows that

\[
    f^{2k}(x) = f^k[f^k(x)] = f^k(x) = x,
\]

so \( x \) is also periodic with period \( 2k \). The same reasoning shows that \( x \) is periodic with periods \( 3k, 4k \), etc. These are not the fundamental period of \( x \). We call the least positive integer for which \( x = f^k(x) \) the prime period of \( x \).

The term prime period can cause some linguistic confusion because prime periods Primality of periods is not the same as primality of need not be prime numbers. It is possible for a function to have a periodic point \( x \) of prime period 4. numbers. This simply means that \( f(x) \neq x, f^2(x) \neq x, f^3(x) \neq x \) but \( f^4(x) = x \).
Let us consider an example. Suppose \( f \) is the function \( f(x) = 1 - x^2 \). What are the fixed points of \( f \)? They are the solutions to the equation \( f(x) = x \), i.e., we solve

\[
1 - x^2 = x, \quad x^2 + x - 1 = 0, \quad x = \frac{-1 \pm \sqrt{5}}{2}.
\]

The graph of the function \( f(x) = 1 - x^2 \). The fixed points of \( f \) are the points of intersection with the line \( y = x \).

To check stability we note that \( f'(x) = -2x \). Now \( f' \) evaluated at these points give \((1 + \sqrt{5}) \approx 3.236\) and \((1 - \sqrt{5}) \approx -1.236\) so \( |f'| > 1 \) at each fixed and the fixed points are unstable.

Notice that \( f(0) = 1 \) and \( f(1) = 0 \), hence 0 and 1 are periodic points of prime period 2. We might wonder if there are other points of prime period 2. Such points must satisfy the equation \( f^2(x) = f(f(x)) = x \). To solve this equation, we first work out a formula for \( f(f(x)) \):

\[
f^2(x) = f(f(x)) \\
= f(1 - x^2) \\
= 1 - (1 - x^2)^2 \\
= 2x^2 - x^4.
\]
The graph of the function $f(x) = 1 - x^2$. The fixed points of $f$ are the points of intersection with the line $y = x$.

Now, we need to solve $f^2(x) = x$, i.e., we solve

$$x = 2x^2 - x^4 \Rightarrow x^4 - 2x^2 + x = 0 \Rightarrow x(x - 1)(x^2 + x - 1).$$

So we find $x = 0$, $x = 1$ and $x = \frac{-1 \pm \sqrt{5}}{2}$ (these are our original fixed points of period one). Thus we find that $x = 0, 1$ are fixed points of period 2.

Now we can ask, Does $f$ have points of prime period 3? If so, they satisfy $f^3(x) = x$ which gives

$$x = f^3(x) = 1 - 4x^4 + 4x^6 - x^8, \Rightarrow 1 - x - 4x^4 + 4x^6 - x^8 = 0.$$ 

This factors a bit to give

$$(1 - x - x^2)(1 + x^2 + x^3 - 2x^4 - x^5 + x^6) = 0.$$ 

Plotting the graph of $g(x) = (1 + x^2 + x^3 - 2x^4 - x^5 + x^6)$ (or using maple) we see that it has no real roots. We can find the six roots of this polynomial by numerical methods, and they are $0.0871062 \pm 0.655455i$, $-1.00914 \pm 0.324759i$, and $1.42203 \pm 0.114188i$. Thus $f$ has no periodic points of prime period 3.

For any function $f$ it is simple in principle to find the points of period $k$. All one has to do is solve the equation $f^k(x) = x$. In practice this can be extremely difficult. If $f(x)$ is
a quadratic polynomial, then the equation \( f^k(x) = x \) is a polynomial of degree \( 2^k \). (When \( k = 10 \), this means finding the roots of a polynomial of degree over 1000.)

**Theorem 2.2.**  1. To find the points of period \( k \), solve the equation \( f^k(x) = x \). Let \( p \) be a point of period \( k \).

2. If \( |(f^k)'(p)| < 1 \), then if the system starts near \( p \), it gravitates to the orbit

\[
\{ p, f(p), f^2(p), \ldots, f^{k-1}(p) \}
\]

This is a stable periodic orbit.

3. Otherwise, if \( |(f^k)'(p)| > 1 \), then

\[
\{ p, f(p), f^2(p), \ldots, f^{k-1}(p) \}
\]

is an unstable orbit, i.e., if the system is started near (but not at) one of these points, subsequent iterations move farther away from the orbit.

**Exercises**

1. For each of the following functions find points of period 1, 2, and 3. Which are prime periodic points? When reasonable, find exact answers; otherwise, use numerical methods. For each periodic point you find, classify it as stable or unstable.

   (i) \( f(x) = 3.1x(1 - x) \)

   (ii) \( f(x) = (-3x^2 + 11x - 4)/2 \)

   (iii) \( f(x) = \cos(x) \)

   (iv) \( f(x) = e^x - 2 \)

   (v) \( f(x) = \frac{1}{3}e^x \)

**2.3.2 Bifurcation for Discrete Systems**

We have studied how to find fixed and periodic points of discrete time dynamical systems \( x(k+1) = f(x(k)) \). We are now interested in gently changing \( f \) and observing what happens
to the fixed and periodic points of its periodic points. We assume that we have a family of functions \( f_a \) where \( a \) is a parameter a number we can adjust. We assume that the function \( f_a \) changes gradually as we change \( a \). In particular, we can think of \( f_a(x) \) as a function of two numbers: \( a \) and \( x \). As such, we require \( f \) to be differentiable with continuous derivatives.

A bifurcation is a sudden change in the number or nature of the fixed and periodic points of the system. Fixed points may appear or disappear, change their stability, or even break apart into periodic points!

**Example 2.1** (Tangent (saddle node) bifurcations). Consider the functions

\[
    f_a(x) = x^2 + a.
\]

We find the fixed points solving

\[
x^2 + a = x \quad \Rightarrow \quad x^2 - x + a = 0, \quad \Rightarrow \quad x = \frac{1 \pm \sqrt{1 - 4a}}{2}.
\]

Notice that if \( a > 1/4 \), then \( f_a \) has no fixed points (because \( f_a(x) = x \) has no real roots). For \( a = 1/4 \) there is a unique fixed point, and for \( a < 1/4 \) there are two fixed points. This can be seen most clearly in the figure.

Graphs of the functions \( f_a(x) = x^2 + a \) for various values of \( a \) near 1/4.

When \( a > 1/4 \) the graph of \( y = f_a(x) \) does not intersect the line \( y = x \) so there are no fixed points. Then, just when \( a = 1/4 \), there is a unique fixed point \( x = 1/2 \). This fixed point is semi-stable (it attracts on the left and repels on the right). Now, just as we decrease \( a \) below 1/4, the fixed point 1/2 splits in two bifurcates. When \( a \) is just below 1/4, the
two fixed points are \((1 \pm \sqrt{1 - 4a})/2\). The larger fixed point is unstable (the curve is steep), while the smaller fixed point is stable. Let’s verify this analytically.

The derivative of \(f_a(x)\) gives \(f'_a(x) = 2x\). At the larger fixed point \(\bar{x}_1 = (1 + \sqrt{1 - 4a})/2\) we have

\[
f'_a(\bar{x}_1) = (1 + \sqrt{1 - 4a}) > 1
\]

confirming \(\bar{x}_1\) is unstable.

At the smaller fixed point \(\bar{x}_2 = (1 - \sqrt{1 - 4a})/2\) we have

\[
f'_a(\bar{x}_2) = (1 - \sqrt{1 - 4a}) < 1
\]

and, so as long as \(a\) is not too much below \(1/4\), it is also greater than \(-1\).

It is interesting to plot both fixed points of \(f_a\) as a function of \(a\).

Bifurcation diagram for \(f_a(x) = x^2 + a\) over the range \(-1/4 \leq a \leq 1/4\)

The horizontal axis represents \(a\), and the vertical axis is \(x\). For each value of \(a\) we plot the fixed points of \(f_a\). Notice that to the right of \(a = 1/4\) there are no fixed points, then as \(a\) decreases, we suddenly have a unique semistable fixed point at \(a = 1/4\) which splits in two below \(1/4\). This sudden change in fixed point behavior is called a bifurcation. This particular example (with the sudden appearance and then splitting of a fixed point) is called a tangent (or saddle node) bifurcation. It is called a tangent bifurcation because the curves \(y = f_a(x)\) become tangent to the line \(y = x\) at the bifurcation value (in this example \(1/4\)).

**Example 2.2** (Period-doubling (pitchfork) bifurcations). We now let’s see what happens near \(a = -3/4\). The larger fixed point \(\bar{x}_1 = (1 + \sqrt{1 - 4a})/2\) satisfies \(f'(\bar{x}_1) = (1 + \sqrt{1 - 4a}) > 1\)
and so is unstable. The other fixed $\bar{x}_2 = (1 - \sqrt{1-4a})/2$ has $f'_a(\bar{x}_2) = (1 - \sqrt{1-4a})$.

When $a > -3/4$ we have $|f'_a(\bar{x}_2)| < 1$ so $\bar{x}_2$ is stable. However, when $a < -3/4$ we have $f'_a(\bar{x}_2) < -1$ and therefore $\bar{x}_2$ becomes unstable.

As an example, if we fix $a = -.8 < -3/4$ then $\bar{x}_2 \approx -0.5246951$ so we take an initial condition $x_0 = -.5$ close to $\bar{x}_2$ and plots the first 100 iterations. Notice that the values appear to be periodic with period 2.

Several iterations of $f_a(x) = x^2 + a$ with $a = -.8 < -3/4$, starting at $x_0 = -0.5$, which is near $\bar{x}_2$.

To understand this phenomenon, it helps to first find the points of period 2 for $f_a$. In other words, we need to solve the equation

$$f_a(f_a(x)) = x$$

Thus we have

$$x = f_a(f_a(x)) = f_a(x^2 + a) = (x^2 + a)^2 + a = x^4 + 2ax^2 + a^2 + a.$$

or

$$x^4 + 2ax^2 - x + a^2 + a = 0.$$  

This factors to

$$(x^2 - x + a)(x^2 + x + a + 1) = 0.$$  

Thus there are four roots of $f_a^2(x) = x$.

$$
\frac{(1 \pm \sqrt{1 - 4a})}{2}, \quad \frac{(-1 \pm \sqrt{-3 - 4a})}{2}.
$$

The first two fixed points are the ones we already know about of period 1.

The other two roots, which we call $p_1$ and $p_2$, are therefore points of prime period 2. We need the term under the square-root sign to be positive in order for these to be real roots, so we need

$$-3 - 4a \geq 0 \Rightarrow a \leq 3/4.$$  

Note that when $a = -3/4 \ p_1 = p_2 = \bar{x}_2 = -1/2$. Just as the point $\bar{x}_2$ goes from stable to unstable it gives birth to a pair of points of period 2.

Bifurcation diagram for $f_a(x) = x^2 + a$ showing the pitchfork bifurcation at $a = -3/4$.

Notice that as $a$ drops past $-3/4$ we see two new curves in the bifurcation diagram. The pitchfork has three branches: the middle is the fixed point $\bar{x}_2$ and the other two are $p_1$ and $p_2$.

We can check the stability of these points of period 2. We need to check $|(f_a^2)'(x)|$ when
$x = p_1, p_2$. By the chain rule

$$(f_a^2)'(x) = f'_a(f_a(x)) \cdot f'_a(x).$$

So we have

$$(f_a^2)'(p_1) = f'_a(f_a(p_1)) f'_a(p_1) = f'_a(p_2) f'_a(p_1) = 2p_2 \cdot 2p_1 = 4 + 4a.$$  

Similarly,

$$(f_a^2)'(p_2) = 4 + 4a.$$  

When $a < -3/4$ we see that $4 + 4a < 1$ and as long as $a > -5/4$ we have $4 + 4a > -1$. Thus for $-5/4 < a < -3/4$ we know that the periodic points $p_1$ and $p_2$ are stable.

This bifurcation where a stable fixed point becomes unstable and casts off two stable points of period 2 – is called a pitchfork or period-doubling bifurcation.

In our examples $(f_a(x) = x^2 + a)$ we saw that at $x = 1/4$ there was a sudden appearance of two fixed points: the unstable $\pi_1$ and the stable (for the moment) $\pi_2$. Then as $a$ drops through $-3/4$ the fixed point $\pi_2$ becomes unstable and breaks apart into two stable points of period 2: $p_1$ and $p_2$. We stop our explicit analysis here but we point out that as $a$ drops through $-5/4$ these points becomes unstable and give rise to four points of period 4. What happens next? Not surprisingly, as $a$ drops a bit farther these four points destabilize and give rise to a stable orbit of period 8. As a drops a tiny bit more the eight points of period 8 bifurcate again to give 16, then 32, etc. These bifurcations become increasingly hard to compute exactly, so we switch to numerical methods.

It is not hard to write a simple computer program to do the computations. What you do see are the stable periodic points of the system for your chosen value of $a$. What you dont see are the unstable ones. We can do this for several values of $a$ and then plot a graph. On the horizontal axis we record the values of $a$, and on the vertical axis we plot the periodic points the computer finds. The branches of subsequent splits into orbits of period 16, 32, 64, etc., are too tightly clustered to see. After all the period-doubling has happened (somewhere around $a = -1.4$), we enter a chaotic region. For some values between $a = -1.5$ and $a = -2$
we have periodic behavior. For example, just below $a = -1.75$ it looks like we have an attractive orbit of period 3.

**Example 2.3 (Transcritical bifurcations).** Before we leave this section, we consider one more type of bifurcation: the transcritical bifurcation. To illustrate the transcritical bifurcation, we use a different family of functions: Let $g_a(x) = x^2 + ax$.

Plots of the function $g_a(x) = x^2 + ax$ for various values of $a$.

To find the fixed points, we solve the equation $g_a(x) = x$, i.e.,

$$x^2 + ax = x \Rightarrow x^2 + (a - 1)x = x[x + (a - 1)] = 0$$

so we get fixed points $x = 0$ (for all $a$) and $x = 1 - a$. Notice that at the special value $a = 1$ these two fixed points become one. Something interesting is happening there! To check the stability of these fixed points we notice that $g'_a(x) = 2x + a$. Now

$$g'_a(0) = a \quad \text{and} \quad g'_a(1 - a) = 2 - a.$$  

When $-1 < a < 1$, we see that 0 is a stable fixed point and $(1 - a)$ is unstable, but in the interval $1 < a < 3$ we have $(1 - a)$ is stable and 0 is unstable. At the value $a = 1$ they swap roles. Notice that at the bifurcation value $a = 1$ the two fixed points merge, and when
they split apart and they have swapped stability.

This is a transcritical bifurcation: two fixed points that merge and then split apart.

### 2.4 Chaos

In the last section we explored the family of functions \( f_a(x) = x^2 + a \). In this section we consider two particular values of \( a \), the case \( a = -1.95 \), and the case \( a = -2.64 \). The particular values are not especially important. What is important is that \(-1.95\) is just slightly greater than \(-2\) and that \(-2.64\) is less than \(-2\). Suppose \( a > -2 \) and we iterate \( f_a \). If the initial value is in the interval \([-2, 2]\), then the iterations stay within \([-2, 2]\) as well. When \( a < -2 \), we will see that for most values \( x \), the iterates \( f^k(x) \) tend to infinity. The set of values \( x \) for which \( f^k(x) \) stays bounded is quite interesting, and the behavior of \( f \) on that set can be worked out precisely.

We perform numerical experiments with the function \( f(x) = x^2 - 1.95 \). We compute the first 1000 iterations of \( f \), starting with initial value \( x = 0.5 \), i.e., we compute

\[
f(-0.5), f^2(-0.5), f^3(-0.5), \ldots, f^{1000}(-0.5)
\]

We graph the first 100 iterates

![Graph of one hundred iterations](image)

One hundred iterations of \( f(x) = x^2 - 1.95 \) starting with \( x = -0.5 \).

The iterations do not seem to be settling down into a periodic behavior but the values
remain bounded. The iterations continue in what appears to be a random pattern. Of course, the pattern isn’t random at all! The numbers are generated by a simple deterministic rule, \( f(x) = x^2 - 1.95 \).

Next we repeat the experiment only this time starting with \( x = -0.50001 \) which is very near \(-.5\). So we compute

\[
f(x), \ f^2(x), \ f^3(-.5), \ldots, \ f^{1000}(x)
\]

then we plot the value of the difference between what we get these two values for \( x \). We would expect that the values of the iterates will stay close since the initial values are close.

![Graph of the difference between \( f^k(-.5) \) and \( f^k(-.500001) \) for \( k = 1 \) to \( 100 \).]

The difference \( f^k(-.5) - f^k(-.500001) \) for \( k = 1 \) to \( 100 \).

The first dozen or so are numerically very close to the iterations we computed starting with \( x = -.500001 \) remain close to those with \( x = -.5 \) but after that they move apart.

Subtle differences in \( x \) lead to enormous differences in \( f^k(x) \). We are witnessing sensitive dependence on initial conditions.

Several views of the bifurcation diagram for \( f_a(x) = a + x^2 \) is plotted in the next three figures.
Bifurcation diagram for $-2 < a < 0$.


Bifurcation diagram for $-2 < a < -3/4$.

Exercises
1. For each of the following families of functions $f_a$ find values $a$ at which the family undergoes bifurcations. Categorize the bifurcations you find (as saddle node, etc.).

   (i) $f_a(x) = ae^x$
   (ii) $f_a(x) = a\sin(x)$
   (iii) $f_a(x) = \sin(ax)$
   (iv) $f_a(x) = a + 2\cos(x)$
   (v) $f_a(x) = e^{a-x^2}$

[Note: It is helpful to plot several members of family of functions $f_a(x)$ on the same set of axes.]

2. The function $f_a(x) = ax(1-x)$ is called the logistic map.

   (a) Write a program to generate a bifurcation diagram for the family $f_a(x)$.

   (Note: You will need to do some exploring to figure out the right range of values for the parameter $a$.)

   (b) Find an exact value of $a$ at which there is a tangent bifurcation.

   (c) Find two exact values of $a$ at which there is a period doubling bifurcation.

2.5 Complex Dynamical Systems

2.5.1 Julia Sets

Up to this point in our work we have been using real numbers. We now invite the return of complex numbers to our study of dynamical systems. We will explore discrete time dynamical system in one complex variable. In other words, we ask, What happens when we iterate a function $f(z)$ where $z$ may be a complex number? In several previous sections we considered the family of functions $f_a(x) = x^2 + a$.

For this family we define

$$B_a = \{ z : |f_a^k(z)| \not\to \infty \text{ as } k \to \infty \}$$
and
\[ U_a = \{ z : |f_a^k(z)| \to \infty \text{ as } k \to \infty \} \]

Note that \( B_a \) and \( U_a \) are complementary sets. The boundary between these sets is denoted \( J_a \). The set \( J_a \) is called the Julia set of the function \( f_a \), and the set \( B_a \) is called the filled-in Julia set of \( f_a \).

We can make a picture of the set \( B_a \) as follows. Every complex number \( z \) corresponds to a point in the plane. We can plot a point for every element of \( B_a \) and thereby produce a two-dimensional depiction of \( B_a \).

The set \( B_{-3/4} \) is symmetrical with respect to both the real \((x)\) and the imaginary \((y)\) axes. It runs from \(-1.5\) to \(1.5\) on the real axis and roughly between \(\pm0.9\) on the imaginary.

![Filled-in Julia set \( B_a \) for \( a = -3/4 \).](image)

Notice that \( B_{-3/4} \) is a rather bumpy set. These bumps dont go away as we look closer. In the following figure we greatly magnify the bump attached to the upper right part of the main section of \( B_{-3/4} \). Notice that we see the same structure as we do in the whole. The set is a fractal!
A close-up of one of the bumps on the main section of $B_{-3/4}$.

What do other Julia sets look like?

The filled-in Julia set $B_{-0.85+0.18i}$.

**Exercises**

1. What is the filled-in Julia set $B_0$? Hint: You do not need a computer.

2. Consider the set $B_{-6}$. Show that $2 \in B_{-6}$. Find several other values in $B_{-6}$

3. Find some points in $B_{-1+3i}$.
2.5.2 The Mandelbrot set

If you create a variety of Julia sets (either using your own program, or using one of the many commercial and/or public-domain packages available), you may note that for some values of $a$ the set $B_a$ is fractal dust, and for some values of $a$ the set $B_a$ is a connected region.

There is a simple way to decide which situation you are in: Iterate $f_a$ starting at 0. If $f_a^k(0)$ remains bounded, then the set $B_a$ will be connected, but if $|f_a^k(0)| \to \infty$ then $B_a$ will be fractal dust. The justification of this fact is beyond the scope of our class.

This leads to a natural question, For which values $a$ does $f_a^k(0)$ remain bounded and for which values of $a$ does it explode? The Mandelbrot set, denoted by $\mathcal{M}$, is the set of values $a$ for which $f_a^k(0)$ remains bounded, i.e.,

$$\mathcal{M} = \{a \in \mathbb{C} : |f_a^k(0)| \not\to \infty\}.$$

For complex values of $a$ it might be reasonable to expect that $\mathcal{M}$ has a simple appearance. Instead, we are startled to see that $\mathcal{M}$ looks like the image in Figure

The Mandelbrot set $\mathcal{M}$. 